Bull. Korean Math. Soc. **21**(1984), No. 1, 42-44

ON THE CONVERGENCE OF DISCRETE TIME SUBMARTINGALES

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This thesis is concerned with the convergence of discrete time submartingales which are not L^1-bounded.

The following example is our main results in this paper. The martingale X in the example is L^1-bounded but it is not uniformly integrable. The martingale $Y=T(X)$ in the example is not L^1-bounded but it converges a.s. to a finite $r.v.$ $Y_\infty=T_\infty(X)$ but $Y^*=T^*(X)<\infty$ a.s. and $S(Y)<\infty$ a.s.

Example 1. Let $\Omega=[0,1]$, $\mathcal{F}$ the σ-field of the Lebesque measurable subsets of $[0,1]$ and P the Lebesque measure on $\mathcal{F}$. Define for $n\geq 1$,

$$X_n=2^n I\left[\frac{2^n-1}{2^n},\ 1\right],$$

$$\mathcal{F}_n=\sigma[X_1,\ X_2,\ \cdots,\ X_n]$$

$$=\sigma\left[\left[0,\ \frac{1}{2}\right),\ \left[\frac{2^n-1}{2^n},\ 1\right],\ \left[\frac{2^k-1}{2^k},\ \frac{2^{k+1}-1}{2^{k+1}}\right) \text{ for } 1\leqslant k\leqslant n-1\right]$$

then $X=(X_n, \mathcal{F}_n, n\geqslant 1)$ is a martingale.

Let $\mathcal{F}_0=\mathcal{F}_1$, $\nu_1=X_1$ and $\nu_n=X_{n-1}$ $(n\geqslant 2)$. Then $\nu=(\nu_n, \mathcal{F}_n, n\geqslant 1)$ is a predictable process and the Burkholder transform of X by ν, $Y=T(X)$ is defined by

$$T_n(X)=\sum_{k=2}^{n} X_{k-1}(X_k-X_{k-1})+X_1^2\ (n\geqslant 2),$$

$$T_1(X)=v_1X_1=X_1^2.$$

Since

$$X_{n-1}(X_n-X_{n-1})=2^{2(n-1)}I\left[\frac{2^n-1}{2^n},\ 1\right]-2^{2(n-1)}I\left[\frac{2^{n-1}-1}{2^{n-1}},\ \frac{2^n-1}{2^n}\right)$$

for $n\geq 2$, we obtain that for $n\geq 2$,

$$T_n(X)=T_{n-1}(X)+2^{2(n-1)}I\left[\frac{2^n-1}{2^n},\ 1\right]-2^{2(n-1)}I\left[\frac{2^{n-1}-1}{2^{n-1}},\ \frac{2^n-1}{2^n}\right).$$

Thus we obtain

$$T_1(X)=2^2 I\left[\frac{1}{2},\ 1\right],$$

$$T_2(X)=(2^2+2^2)I\left[\frac{2^2-1}{2^2},\ 1\right],$$

Received 29 December 1983. Thesis submitted to Korea University, December 1981. Degree approved February 1982. Supervisor: Professor Yu, Hi-Sé.

$$T_n(X)=\left(\frac{2^{2n}+2^3}{3}\right)I\left[\frac{2^n-1}{2^n},\ 1\right]-\sum_{k=2}^{n-1}\left(\frac{2^{2k+1}-2^3}{3}\right)I\left[\frac{2^k-1}{2^k},\ \frac{2^{k+1}-1}{2^{k+1}}\right).$$

for $n\geq 3$. Since $v=(v_n,\ \mathcal{F}_n,\ n\geq 1)$ is a predictable process in L^∞, $Y=T(X)=(T_n(X),\ \mathcal{F}_n,\ n\geq 1)$ is also a martingale.

This example also suggests us the following theorems which are some generalized results in [1], [2], [3], [4] and [5].

THEOREM 1. *Let X be an L^1-bounded submartingale and v^* finite a.s. Then there does not exist a positive constant M which is independent on X and on the probability space such that for every $c>0$,*

$$cP\{T^*(X)>c\}\leq M\|X\|_1.$$

THEOREM 2. *Under the same conditions in Theorem 1, there is no constant M_X depending on X such that for every $c>0$,*

$$cP\{T^*(X)>c\}\leqslant M_X\|X\|_1.$$

THEOREM 3. *Let X be an L^1-bounded submartingale and $v^*\in L^\infty$. Then there is a constant M_X depending on X such that for every $c>0$,*

$$cP\{T^*(X)>c\}\leqslant M_X\|X\|_1.$$

THEOREM 4. *Let $(X_n,\ n\in N)$ be an L^1-bounded submartingale. Then for any stopping time T, X_T is integrable.*

THEOREM 5. *Let $(X_n,\ n\in N)$ be an L^1-bounded nonnegative increasing submartingale. Then the transform $(T_n(X),\ n\in N)$ converges a.s. on $\{v^*<\infty\}$.*

THEOREM 6. *Let $(X_n,\ n\in N)$ be an L^1-bounded submartingale. Then the transform $(T_n(X),\ n\in N)$ converges a.s. on $\{v^*<\infty\}$.*

THEOREM 7. *Let $(X_n,\ \mathcal{F}_n,\ n\in N)$ be an L^1-bounded submartingale. Then its sqare function $S(X)=\lim_n(\sum_{k=1}^{n}x_k^2)^{\frac{1}{2}}$ is finite a.s.*

THEOREM 8. *If X is an L^1-bounded submartingale, then there is a constant M_X depending on X such that, for every $c>0$,*

$$cP\{S(X)>c\}\leqslant M_X\|X\|_1.$$

THEOREM 9. *Let $X=(X_n,\ \mathcal{F}_n,\ n\in N\}$ be an L^1-bounded submartingale and $Y=(Y_n,\ \mathcal{F}_n,\ n\in N)$ a martingale. If $S_n(Y)\leq S_n(X)$, $n\in N$, then Y converges a.s.*

THEOREM 10. *Let Y be a process. If there is an L^1-bounded submartingale X such that the transform of X, $Y=T(X)$ by a predictable process v with $v^*<\infty$ a.s., then $S(Y)$ is finite a.s.*

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