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ON A PROPERTY OF CONVOLUTION OPERATORS IN THE SPACES D'_{L^p}, $p\geqslant 1$ AND $\mathcal{S}'$

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Let D'_{L^p} be the space of distributions of L^p-growth and $\mathcal{S}'$ the space of tempered destributions in $\mathbf{R}^n$: D'_{L^p}, $1\leqslant p\leqslant\infty$, is the dual of the space D_{L^p} which we discribe later. We denote by $O_c'(\mathcal{S}':\mathcal{S}')$ the space of convolution operators in $\mathcal{S}'$.

In [8] S. Sznajder and Z. Zielezny proved the following necessary conditions for convolution operators in $O_c'(\mathcal{S}':\mathcal{S}')$ to be solvable in $\mathcal{S}'$.

THEOREM [8]. *If S is a distribution in $O_c'(\mathcal{S}':\mathcal{S}')$ and $\hat{S}$ is its Fourier transform, then the following conditions are equivalent:*

(a) *For every integer k there exists an integer $m\geqslant 0$ and constants μ, $M\geqslant 0$ such that*

$$\sup_{|\alpha|\leqslant m,\ s\in\mathbf{R}^n,\ |s-\xi|\leqslant(1+|\xi|)^{-k}} |D^\alpha\hat{S}(s)|\geqslant|\xi|^{-\mu},\ \textit{where}\ \xi\in\mathbf{R}^n,\ |\xi|\geqslant M.$$

(c) *If $u\in O_c'(\mathcal{S}':\mathcal{S}')$ and $S*u\in\mathcal{S}$, then $u\in\mathcal{S}$.*

Conditions (a) and (c) follows from

(b) $S*\mathcal{S}'=\mathcal{S}'$.

However, the converse implication is not true since condition (a) does not restrict the order of zeros of $\hat{S}$. In this connection they made the following conjecture.

Conjecture. If S is a distribution in $O_c'(\mathcal{S}':\mathcal{S}')$ and the order of the zeros of its Fourier transform $\hat{S}$ is bounded, then the conditions (a) and (c) are equivalent to (b).

In connection with this sufficiency question, we consider the operators in $O_c'(\mathcal{S}':\mathcal{S}')$ whose Fourier transforms do not have any zeros. Of course, this kinds of operators have "much stronger property" than that suggested in above conjecture. But we have found these operators are very interesting for the problem of solvability in $\mathcal{S}'$ and D'_{L^p}, $1\leqslant p\leqslant\infty$.

In this paper we show that the property (7) in [2] is a sufficient condition for operators in $O_c'(\mathcal{S}':\mathcal{S}')$ with no zeros to be solvable in $\mathcal{S}'$. Furthermore, this property is a necessary and sufficient condition for operators in $O_c'(\mathcal{S}':\mathcal{S}')$ to be solvable in D'_{L^1}.

Before presenting our results we recall the basic facts about the spaces D'_{L^p}, $1\leqslant p\leqslant\infty$ and $\mathcal{S}'$. For the proof we refer [1, 3, 5].

The space D'_{L^p}, $1\leqslant p\leqslant\infty$. Let D_{L^p} be the space of all C^∞-functions ϕ in $\mathbf{R}^n$ such that $D^\alpha\phi$, for all $\alpha\in\mathbf{R}^n$, is in $L^p(\mathbf{R}^n)$ equipped with the topology generated by countable norms

$$||\phi||_{m,p}=\{\sum_{|\alpha|\leq m}||D^\alpha\phi||^p_{L^p}\}^{1/2},\ m\in N.$$

Then it is obviously a Frechet space and a normal space of distributions in $\mathbf{R}^n$. We also have $C_c^\infty\subset D_{L^p}\subset D'$ with continuous injections.

We denote by $D'_{L^p}, 1\leqslant p\leqslant\infty$, the dual of D_{L^q}, where $\frac{1}{q}+\frac{1}{q}=1$ and these duals are subspaces of the space of distributions in $\mathbf{R}^n$. A distribution T is in D'_{L^p}, $1\leqslant p\leqslant\infty$, if and only if there is an integer $m(T)>0$ such that

(1) $$T=\sum_{|\alpha|\leq m}D^\alpha f_\alpha,\quad \alpha\in\mathbf{N}^n$$

where the f_α's are bounded continuous functions belonging to $L^p(\mathbf{R}^n)$. Moreover, if $p<\infty$, each f_α converges to zero at infinity.

The Fourier transform of a function in D_{L^1} is a continuous function rapidly decreasing at infinity and also the Fourier transform of a distribution in D'_{L^1} is a continuous function slowly increasing at infinity.

The space $\mathcal{S}'$. Let $\mathcal{S}$ be the space of all C^∞-functions ϕ in $\mathbf{R}^n$ such that

$$\sup_{|\alpha|\leqslant k, x\in\mathbf{R}^n}(1+|x|)^k|D^\alpha\phi(x)|<\infty,\quad k=0,1,2,\ldots$$

equipped with the topology generated by these countable norms. We denote by $\mathcal{S}'$ the dual of $\mathcal{S}$. The Fourier transformation is now an isomorphism of $\mathcal{S}$ onto itself and of $\mathcal{S}'$ onto $\mathcal{S}'$.

The space $O_c'(\mathcal{S}':\mathcal{S}')$ of convolution operators in $\mathcal{S}'$ consists of distributions $S\in\mathcal{S}'$ satisfying one of the following equivalent conditions:

(i) Given any $k=0,1,2,\ldots$, S can be represented in the form

(2) $$S=\sum_{|\alpha|\leq m}D^\alpha f_\alpha,$$

where f_α, $|\alpha|\leqslant m$, are continuous functions in $\mathbf{R}^n$ such that

$$f_\alpha(x)=0((1+|x|)^{-k})\ \text{as}\ |x|\to\infty.$$

(ii) For every ϕ in $\mathcal{S}$, $S*\phi$ is in $\mathcal{S}$. Moreover, the mapping $\phi\longrightarrow S*\phi$ of $\mathcal{S}$ into $\mathcal{S}$ is continuous.

The Fourier transform $\hat{S}$ of a distribution S in $O_c'(\mathcal{S}':\mathcal{S}')$ is a C^∞-function with the following property: For every multi-index α there exists a non-negative integer l such that

(3) $$D^\alpha\hat{S}(\xi)=0((1+|\xi|)^l)\ \text{as}\ |\xi|\to\infty.$$

We denote by $O_M(\mathcal{S}':\mathcal{S}')$ the space of all C^∞-functions with the above property (3). They are multiplication operators in $\mathcal{S}'$ and the Fourier transformation is an isomorphism of $O_c'(\mathcal{S}':\mathcal{S}')$ onto $O_M(\mathcal{S}':\mathcal{S}')$ (see [4] vol. II).

We now prove a sufficient condition mentioned early, for operators in $O_c'(\mathcal{S}':\mathcal{S}')$ to be solvable in $\mathcal{S}'$.

THEOREM 1. *Let S be a operator in $O_c'(\mathcal{S}':\mathcal{S}')$ whose Fourier transform has no zeros. If there are constants a, M such that*

(4) $$|\hat{S}(\xi)|\geqslant|\xi|^a\ \textit{for all}\ \xi\in\mathbf{R}^n\ \textit{and}\ |\xi|\geqslant M,$$

then S is solvable in D'_{L^p}, $1 \leqslant p \leqslant \infty$ and in $\mathcal{S}'$. where S is solvable in D'_{L^p} (in $\mathcal{S}'$) means that $S*D'_{L^p}=D'_{L^p}$ (resp. $S*\mathcal{S}=\mathcal{S}$).

Proof. From the facts that $\hat{S}$ is in $O_M(\mathcal{S}' : \mathcal{S}')$ and has no zeros in $\mathbf{R}^n$, we can easily show that $\hat{S}^{-1}$ is in $O_M(\mathcal{S}' : \mathcal{S}')$ using (4). Since the Fourier transformation is an isomorphism of $O_c'(\mathcal{S}' : \mathcal{S}')$ onto $O_M(\mathcal{S}' : \mathcal{S}')$, there is a convolution operator S_1 in $O_c'(\mathcal{S} : \mathcal{S}')$ whose Fourier transform $\hat{S}_1$ is identically equal to $\hat{S}^{-1}$. Then we have

(5) $$S \hat{*} S_1 = \hat{S}\hat{S}_1 = 1 = \hat{\delta}, \text{ that is,}$$
$$S*S_1=\delta.$$

For any given v in D'_{L^p} (resp., in $\mathcal{S}'$), S_1*v is in D'_{L^p} since $O_c'(\mathcal{S}' : \mathcal{S}')$ is a subspace of D'_{L^1}. Therefore, applying (5) we have

$$S*(S_1*v)=(S*S_1)*v=\delta*v=v.$$

This proves that S is solvable in D'_{L^p}, $1 \leqslant p \leqslant \infty$ and is $\mathcal{S}'$.

We next have the following partial converse result using the theorem 2 in [2].

THEOREM 2. *Let S be an operator in $O_c'(\mathcal{S}' : \mathcal{S}')$ satisfying the condition* (4). *If S is solvable in D'_{L^∞}, then its Fourier transform $\hat{S}$ has no zeros.*

Proof. We can take an element v in $D_{L^1} \subset D'_{L^\infty}$ whose Fourier transform $\hat{v}$ has no zero in $\mathbf{R}^n$. By the solvability of S in D'_{L^∞} there is an element u in D'_{L^∞} such that

(6) $$S*u=v.$$

In view of the theorem 2 in [2], the condition (4) implies that u is in D_{L^2}. Taking Fourier transform in (6) we have

(7) $$S \hat{*} u = \hat{S}\hat{u} = \hat{v}.$$

From the facts that $\hat{u}(\xi)$ is a continuous function with no zero in $\mathbf{R}^n$ and $\hat{u}(\xi)$ is the product of a polynomial and L^2-function, we get our result in counting that $\hat{S}(\xi)$ is a C^∞-function in $\mathbf{R}^n$.

In view of conjecture we can see that the condition (4) is too much restricted to be a necessary condition for the solvability of S in $\mathcal{S}'$. But we have found a subspace of the space of distributions in $\mathbf{R}^n$ in which the property (4) is a necessary and sufficient condition for the solvability of the given convolution operator.

THEOREM 3. *Let S be a operator in $O_c'(\mathcal{S}' : \mathcal{S}')$. Then S is solvable in D'_{L^1} if and only if its Fourier transform has no zero in $\mathbf{R}^n$ and it satisfies the condition* (4) *for some constants M and a.*

Proof. Sufficiency is already done in Theorem 1. From the solvability of S in D'_{L^1}, there is a distribution E in D'_{L^1} such that

(7) $$S*E=\delta.$$

Taking Fourier transformation in both side of (7), we have

(8) $$\hat{S}(\xi)\hat{E}(\xi)=1 \text{ in } \mathbf{R}^n.$$

Since $\hat{S}(\xi)$ and $\hat{E}(\xi)$ are continuous functions in $\mathbf{R}^n$, we have one of our results that $\hat{S}(\xi)$ has no zero in $\mathbf{R}^n$.

For the other result we now assume that S does not satisfy the condition (4). Then there is a sequence $\{\xi_j\}$ in $\mathbf{R}^n$ such that

(9) $$|\xi_j|\to\infty \textit{ as } j\to\infty, \text{ and}$$
$$|\hat{S}(\xi_j)|\leqslant|\xi_j|^{-j} \text{ for } j=1,2,\ldots.$$

On the other hand, from the fact that $\hat{E}$ is in $O_M(\mathcal{S}':\mathcal{S}')$ there are constants μ and C such that

(10) $$|\hat{E}(\xi)|\leqslant C(1+|\xi|)^{\mu} \text{ for all } \xi\in\mathbf{R}^n.$$

Therefore we have the following contradiction:

$$1=|\hat{S}(\xi_j)|\,|\hat{E}(\xi_j)|\leqslant C|\xi_j|^{-j}(1+|\xi_j|^{\mu})\to 0$$

as $j\to\infty$.

We have an example of an operator in $O_c'(\mathcal{S}':\mathcal{S}')$ which satisfies all the requirement in Theorem 3.

EXAMPLE. Taking $S=e^{-|t|}$ in $\mathbf{R}^1$, $\hat{S}(\xi)=\dfrac{1}{1+|\xi|^2}$ satisfies all the conditions in the theorem 3.

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