GLOBAL CONSTANCY PRINCIPLE FOR MIZOHATA OPERATORS

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Introduction

In [1] L. Nirenberg constructed a famous example of a smooth vector field which have only the constant functions as solutions in an open subset of the plane. Explanations of this phenomenon have been proposed in Treves [3] and Sjöstrand [2]. The explanation in [3] is related to the following so-called "local constancy principle".

(L) If a $C^1$ solution $u$ of $Lu=0$ with $du \neq 0$ exists, then, locally, any other $C^1$ solution $h$ of this equation is constant on each set on which $u$ is constant.

In this short paper we shall obtain a sufficient condition for the validity of a global version of the local constancy principle (L) for Mizohata operators $M_k$.

Throughout this paper let $M_k$ ($k$, odd) be the vector field in an open subset $\Omega$ of $R^2$

\[ M_k = \frac{\partial}{\partial y} + iy^k \frac{\partial}{\partial x}. \]

We now give a sufficient condition on $\Omega$ for a Mizohata operator $M_k$ of the form (1) to satisfy the following global version of the local constancy principle (L):

(G) Let $u$ be a $C^1$ solution of

\[ M_k u = 0 \]

in $\Omega$ with $du \neq 0$. Then any other $C^1$ solution $h$ of (2) in $\Omega$ is constant on $u^{-1}(z)$ for every fixed $z \in u(\Omega)$.

Note that there exists a $C^\infty$ function $u$ of (2) in $\Omega$ with $du \neq 0$ for all $\omega \in \Omega$. For example one may take

\[ u = x - iy^{k+1} \frac{1}{k+1}. \]

We denote by $u^+$ and $u^-$ the restrictions of $u$ to $\Omega^+$ and $\Omega^-$, respectively.

Proposition 1. Let $u$ be a solution of (2) in $\Omega$ with $du(\omega) \neq 0$. Then the Jacobians of $u^+$ and $u^-$ are nonvanishing in $\Omega^+$ and $\Omega^-$, respectively.

Proof: Since $\text{Re } u_y = y^k \text{ Im } u_x$ and $\text{Im } u_y = -y^k \text{ Re } u_x$
\[
\frac{\partial (\text{Re } u, \text{ Im } u)}{\partial (x, y)} = -y^2 (\text{Re } u_x)^2 + (\text{Im } u_x)^2
\]
\[
= -\frac{1}{y^4} (\text{Re } u_x)^2 + (\text{Im } u_x)^2.
\]

**Remark.** \(u^\pm (u^-, \text{ resp.})\) in Proposition 1 is a local diffeomorphism from \(Q^\pm (Q, \text{ resp.})\) into \(C\).

Before stating the main theorem, let us examine the following crucial example.

**Example.** Let \(Q\) be the \(S\)-shaped domain in \(R^2\), shown in Fig. 1 and consider

\[
M = \frac{\partial}{\partial y} - iy \frac{\partial}{\partial x} \text{ in } Q.
\]

Take the solution \(u = x + iy^2/2\) of \(Mu = 0\).

![Fig. 1](image)

Clearly \(du\) does not vanish in \(Q\). Then \(u\) maps onto the annulus shaped domain in \(C\) in Fig. 1. Put

\[
u(Q^+) \cap u(Q^-) = A \cup B
\]

with \(A, B\) indicated in Fig. 1. Note that the closure of \(A\) contains the line segment \(\Gamma\) which is the image of \(\Omega_0 = \{(x, y) \in \Omega | y = 0\}\) under the mapping \(u\).

Let \(h\) be any other \(C^1\) solution of \(Mu = 0\) and let \(h^\pm\) be the restriction of \(h\) to \(Q^\pm\). Then we can form \(\tilde{h}^\pm = h^\pm \circ u^{-1}\), since \(u^\pm\) is a diffeomorphism of \(Q^\pm\) onto \(u(Q^\pm)\). Thus \(\tilde{h}^\pm\) is defined in \(u(Q^\pm)\). Since \(d(h^\pm du) = -(Mh^\pm) dx \wedge dy = 0\) in \(Q^\pm\), we have \(d(\tilde{h}^\pm dz) = 0\) in \(u(Q^\pm)\). In other words, \(h^\pm\) is holomorphic in \(u(Q^\pm)\). As \(z\) approaches any point in \(\Gamma\) \(\tilde{h}(z) = h^+(z) - h^-(z)\) tends to zero. Therefore by analytic continuation we must have \(\tilde{h}(z) = 0\), i.e., \(h^+(z) = h^-(z)\) in \(A\). However, \(\tilde{h}^+\) and \(\tilde{h}^-\) do not have to coincide in \(B\). Examples of this type can be easily constructed.
This example shows us that the connectedness of \( u(\Omega^+) \cap u(\Omega^-) \) is crucial for (G) to be true. We now prove that this condition is sufficient for (G).

**Theorem 1.** Let \( \Omega \) be an open subset of \( \mathbb{R}^2 \) and \( u \) be a solution of (2) with \( du \neq 0 \) in \( \Omega \), \( u^+ \) and \( u^- \) injective in \( \Omega^+ \) and \( \Omega^- \), respectively. Suppose that \( u(\Omega^+) \cap u(\Omega^-) \) is connected. Then (G) holds for \( M_k \).

**Proof.** Put \( G = u(\Omega^+) \cap u(\Omega^-) \). Then the closure of \( G \) contains \( \Gamma = u(\Omega_0) \). Set \( V^\pm = (u^\pm)^{-1}(G) \). Call \( h^\pm \) the restriction of \( h \) to \( V^\pm \). Since \( u^\pm \) is a diffeomorphism of \( V^\pm \) onto \( G \), we put \( \tilde{h}^\pm = h^\pm \circ (u^\pm)^{-1} \) and \( \tilde{h} = \tilde{h}^+ - \tilde{h}^- \) in \( G \). Since \( d(h^\pm du) = -(M_k h^\pm) dx dy = 0 \) in \( V^\pm \) we have \( d(\tilde{h}^\pm dz) = 0 \) in \( G \). Hence \( \tilde{h}^+, \tilde{h}^- \) and \( \tilde{h} \) are holomorphic in \( G \). However \( \tilde{h}(z) \) tends to 0 as \( z \) approaches some point on \( \Gamma \). Since \( \Gamma \) contains a nonempty open arc \( \tilde{h} = 0 \) in \( G \) by analytic continuation. In other words, \( h^\pm u^{-1} = h^- u^{-1} \) in \( G \).

**References**