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GLOBAL CONSTANCY PRINCIPLE FOR MIZOHATA OPERATORS

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Introduction

In [1] L. Nirenberg constructed a famous example of a smooth vector field which have only the constant functions as solutions in an open subset of the plane. Explanations of this phenomenon have been proposed in Treves [3] and Sjöstrand [2]. The explanation in [3] is related to the following so-called "local constancy principle".

(L) If a C^1 solution u of $Lu=0$ with $du\neq 0$ exists, then, locally, any other C^1 solution h of this equation is constant on each set on which u is constant.

In this short paper we shall obtain a sufficient condition for the validity of a global version of the local constancy principle (L) for Mizohata operators M_k.

Throughout this paper let M_k (k, odd) be the vector field in an open subset Ω of R^2

$$M_k=\frac{\partial}{\partial y}+iy^k\frac{\partial}{\partial x}. \tag{1}$$

We now give a sufficient condition on Ω for a Mizohata operator M_k of the form (1) to satisfy the following global version of the local constancy principle (L):

(G) Let u be a C^1 solution of

$$M_k\, u=0 \tag{2}$$

in Ω with $du\neq 0$. Then any other C^1 solution h of (2) in Ω is constant on $u^{-1}(z)$ for every fixed $z\in u(\Omega)$.

Note that there exists a C^∞ function u of (2) in Ω with $du\neq 0$ for all $\omega\in\Omega$. For example one may take

$$u=x-i\frac{y^{k+1}}{k+1}.$$

We denote by u^+ and u^- the restrictions of u to Ω^+ and Ω^-, respectively.

PROPOSITION 1. *Let u be a solution of* (2) *in Ω with $du(\omega)\neq 0$. Then the Jacobians of u^+ and u^- are nonvanishing in Ω^+ and Ω^-, respectively.*

Proof: Since $\operatorname{Re} u_y=y^k \operatorname{Im} u_x$ and $\operatorname{Im} u_y=-y^k \operatorname{Re} u_x$

$$\frac{\partial(\mathrm{Re}\ u,\ \mathrm{Im}\ u)}{\partial(x,y)}=-y^k((\mathrm{Re}\ u_x)^2+(\mathrm{Im}\ u_x)^2)$$
$$=-\frac{1}{y^k}((\mathrm{Re}\ u_y)^2+(\mathrm{Im}\ u_y)^2).$$

REMARK. $u^+(u^-$, resp.) in Proposition 1 is a local diffeomorphism from $\Omega^+(\Omega$, resp.) into C.

Before stating the main theorem, let us examine the following crucial example.

EXAMPLE. Let Ω be the S-shaped domain in R^2, shown in Fig. 1 and consider

$$M=\frac{\partial}{\partial y}-iy\frac{\partial}{\partial x}\text{ in }\Omega.$$

Take the solution $u=x+iy^2/2$ of $Mu=0$.

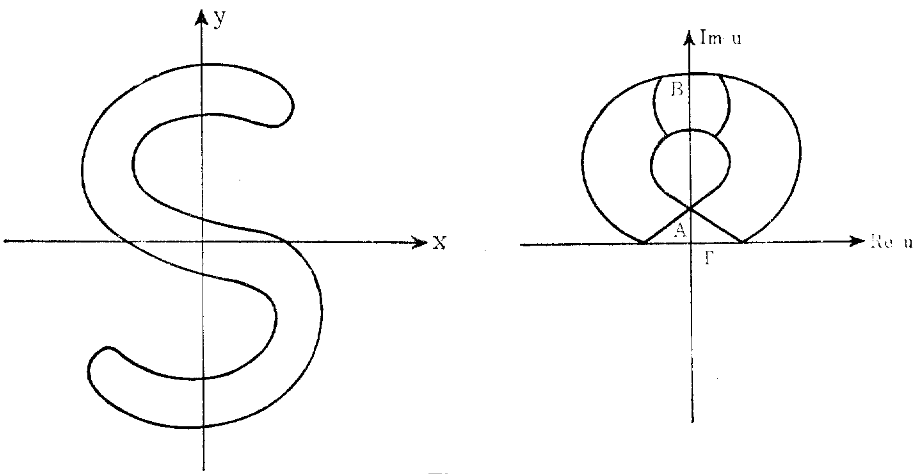

Fig. 1

Clearly du does not vanish in Ω. Then u maps onto the annulus shaped domain in C in Fig. 1. Put

$$u(\Omega^+)\cap u(\Omega^-)=A\cup B$$

with A, B indicated in Fig. 1. Note that the closure of A contains the line segment Γ which is the image of $\Omega_0=\{(x,y)\in\Omega\,|\,y=0\}$ under the mapping u.

Let h be any other C^1 solution of $Mu=0$ and let $h^\pm$ be the restriction of h to $\Omega^\pm$. Then we can form $\tilde{h}^\pm=h^\pm\circ u^{-1}$, since $u^\pm$ is a diffeomorphism of $\Omega^\pm$ onto $u(\Omega^\pm)$. Thus $\tilde{h}^\pm$ is defined in $u(\Omega^\pm)$. Since $d(h^\pm du)=-(Mh^\pm)dx\wedge dy=0$ in $\Omega^\pm$, we have $d(\tilde{h}^\pm dz)=0$ in $u(\Omega^\pm)$. In other words, $h^\pm$ is holomorphic in $u(\Omega^\pm)$. As z approaches any point in Γ $\tilde{h}(z)=h^+(z)-\tilde{h}^-(z)$ tends to zero. Therefore by analytic continuation we must have $\tilde{h}(z)=0$, i.e., $\tilde{h}^+(z)=\tilde{h}^-(z)$ in A. However, $\tilde{h}^+$ and $\tilde{h}^-$ do not have to coincide in B. Examples of this type can be easily constructed.

This example shows us that the connectedness of $u(\Omega^+)\cap u(\Omega^-)$ is crucial for (G) to be true. We now prove that this condition is sufficient for (G).

THEOREM 1. *Let* Ω *be an open subset of* R^2 *and* u *be a solution of* (2) *with* $du\neq 0$ *in* Ω, u^+ *and* u^- *injective in* Ω^+ *and* Ω^-, *respectively. Suppose that* $u(\Omega^+)\cap u(\Omega^-)$ *is connected. Then* (G) *holds for* M_k.

Proof. Put $G=u(\Omega^+)\cap u(\Omega^-)$. Then the closure of G contains $\Gamma=u(\Omega_0)$. Set $V^\pm=(u^\pm)^{-1}(G)$. Call $h^\pm$ the restriction of h to $V^\pm$. Since $u^\pm$ is a diffeomorphism of $V^\pm$ onto G, we put $\tilde{h}^\pm=h^\pm\circ(u^\pm)^{-1}$ and $\tilde{h}=\tilde{h}^+-\tilde{h}^-$ in G. Since $d(h^\pm du)=-(M_k h^\pm)$ $dx\,dy=0$ in $V^\pm$ we have $d(\tilde{h}^\pm dz)=0$ in G. Hence $\tilde{h}^+, h^-$ and $\tilde{h}$ are holomorphic in G. However $\tilde{h}(z)$ tends to 0 as z approaches some point on Γ. Since Γ contains a nonempty open arc $\tilde{h}=0$ in G by analytic continuation. In other words, $h^+\circ u^{-1}=h^-\circ u^{-1}$ in G.

References

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3. F. Treves, *Remarks about certain first order linear PDE in two variables*, Comm. in PDEs, **54** (1980), 381–425.