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CLOSED IDEALS IN A SEMIFINITE, INFINITE VON NEUMANN ALGEBRA, ARISING FROM RELATIVE RANKS OF ITS ELEMENTS

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1. Introduction

Throughout the paper let A be a semifinite, infinite von Neumann algebra acting on a Hilbert space H, α an infinite cardinal. The main purpose of our work is to give several characterizations of a class of closed ideals in A, by introducing the notions of relative ranks of elements in A and the relative α-topology on H. The relative α-topology is an analogue to the α-topology that we have defined in ([7], [8]). The present work is regrded as an extension of [7], [8] and motivated by works of M. Breuer ([1], [2]), V. Kaftal ([5], [6]) and M. G. Sonis [9].

2. Relative rank and ideals

DEFINITION 1. A projection e in a semifinite, infinite von Neumann algebra A is said to have the *relative rank* α, with respect to A, where α is an infinite cardinal, if the following holds: There is a family $\{e_i\}_{i\in I}$ of mutually orthogonal nonzero finite projections in A such that $e=\sum_{i\in I} e_i$, $\text{card}(I)=\alpha$, where $\text{card}(I)$ denotes the cardinality of I. In this situation, the relative rank of e with respect to A is defined to be α and denoted by $\text{rank}_A(e)$ or simply $\text{rank}(e)$.

By (p. 252 [3] Lemma 6), it is clear that $\text{rank}(e)$ is well defined independent of the choice of expressions $\sum_{i\in I} e_i$ of e in Definition 1, and that $\text{rank}(e)=\text{rank}(f)$ whenever e and f are equivalent projections in A. Also one can easily show that $\text{rank}(e)\leq\text{rank}(f)$, if $e\prec f$, $e, f\in P$.

DEFINITION 2. For $x\in A$, let $l(x)$ denote the left support of x in A, that is the range projection of x in A. The *relative rank* of x denoted by $\text{rank}_A(x)$ or simply by $\text{rank}(x)$ is defined to be just $\text{rank}(l(x))$, provided that $l(x)$ has the relative rank of infinite cardinality. When $l(x)$ is finite, we write $\text{rank}(x)<\aleph_0$.

Note that for any $x\in A$, we always have $\text{rank}(x)\leq\dim(H)$. Throughout the

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work, let Γ denote the set of all infinite cardinals α such that $\alpha=\text{rank}(z)$ for some element $z(\in A)$ that has the infinite relative rank.

DEFINITION 3. For every $\alpha\in\Gamma$, we define $I_\alpha=\{x\in A: \text{rank}(x)<\alpha\}$ and $J_\alpha=\bar{I}_\alpha$, the norm closure of I_α in A.

PROPOSITION 1. *For every $\alpha\in\Gamma$, the set I_α is a two sided ideal in A.*

Proof. For $x\in I_\alpha$, $y\in A$, we note that $\text{range}(xy)\subset\text{range}(x)$, so that $xy\in I_\alpha$. Let $l(x)$ denote the left support of x in A. Then, $l(x)\sim l(x^*)$ in A ([10] p. 94 Theorem 4.3). By the remark following Definition 1, we can deduce that rank $l(x^*)=\text{rank}\, l(x)$ $(<\alpha)$. It follows that $x^*\in I_\alpha$, whenever $x\in I$. For x, $y\in I$, we note that $\overline{\text{range}(x+y)}\subset\overline{\text{range}(x)}\vee\overline{\text{range}(y)}$. Let e, f be the left supports of x, y respectively. It suffices to show that $e\vee f\in I_\alpha$. To the third of the next equations, we applied the parallelogram law; $e\vee f-f\sim e-e\wedge f$.

$$\begin{aligned}\text{rank}(e\vee f)&=\text{rank}(f+(e\vee f-f))\\&=\text{rank}\ f+\text{rank}(e\vee f-f)\\&=\text{rank}\ f+\text{rank}(e-(e\wedge f))\\&\leq\text{rank}\ f+\text{rank}\ e\\&<\alpha,\end{aligned}$$

since α is an infinite cardinal, $\text{rank}(e)<\alpha$ and $\text{rank}(f)<\alpha$. It follows that $e\vee f\in I_\alpha$ and that $x+y\in I_\alpha$.

3. Characterization of the closed ideals J_α, $\alpha\in\Gamma$.

The following lemma is a strengthening of a result of R.G. Douglas ([4] p. 413, Theorem 1).

LEMMA 1. *Let a and b be bounded operators on a Hilbert space H. The following statements are equivalent*:

(1) *$\text{range}(a)\subset\text{range}(b)$;*
(2) *$aa^*\leq\lambda^2bb^*$ for some $\lambda\geq0$; and*
(3) *there is a bounded operator c on H so that $a=bc$.*

Moreover, if any one of (1), (2) *and* (3) *are valid, then there exists a unique operator c so that*

(a) $\|c\|^2=\inf\{\mu: aa^*\leq\mu bb^*\}$
(b) *$\text{null}(a)=\text{null}(c)$; and*
(c) $\text{range}(c)\subset\overline{\text{range}(b^*)}$. *This unique operator c lies in the von Neumann algebra generated by a and b.*

Proof. Let $R=R(a,b)$ be the von Neumann algebra generated by a and b. If one examines the original proof of ([4] p. 413, Theorem), especially that for (2) $\rightarrow$ (3) ([4] p. 414), the unique c is defined as follows.
First define d by $d(b^*\xi)=a^*\xi$, for every $\xi\in H$ and $d(b^*(H)^\perp)=0$, through

the obvious extension as a bounded operator on H. We then put $c=d^*$. By using the double commutant theorem ([10] p. 69, Corollary 3.3), we would like to show that $c\in R$.

For every $t\in R(a,b)$ and every $\xi\in H$, we have

$$tdb^*\xi=ta^*\xi,$$
$$dtb^*\xi=db^*t\xi=a^*t\xi=ta^*\xi$$

Consequently, $td(b^*\xi)=dt(b^*\xi)$. Hence, $(td)|\overline{b^*(H)}=(dt)|\overline{b^*(H)}$. Now let $\eta\in B^*(H)$. By the way that d was defined in the Douglas paper ([4] p. 414), we see that $d\eta=0$. On the other hand, $t(b^*(H))=b^*(t(H))\subset b^*(H)$, $(t^*(b^*(H))=b^*(t^*(H))\subset b^*(H)$. Consequently, both $\overline{b^*(H)}$ and $b^*(H)^\perp$ are invariant under t. It follows that $t\eta\in b^*(H)^\perp$ and that $dt\eta=0=td\eta$. This shows that $(td)|b^*(H)^\perp=(dt)|b^*(H)^\perp$. Consequently, $dt=td$ for all $t\in R(a,b)'$. Hence $d\in R(a,b)''=R$, as desired.

LEMMA 2. *Let K and H be two Hilbert spaces. Let U be the set of all bounded operators from K into H which are bounded below. Then U is an open subset of the set $B(K,H)$ of all bounded operators from K into H, with respect to the norm.*

Proof. Let a be an arbitrary element of U. Let us put $b=a^*a$. Then b is an element of the set $B(K)$ of all bounded operators from K into itself, and in fact b is also bounded below. This means that b is left invertible in $B(K)$. Since b is selfadjoint it is in fact an invertible element in $B(K)$. Since the set G of all invertible elements in $B(K)$ is an open set with respect to the norm we can say that there is a norm open neighborhood V of b in G. Since $x\in B(K,H)\to x^*x\in B(K)$ is norm continuous, there is an open neighborhood W of a in $B(K,H)$ such that whenever $c\in W$, $c^*c\in V$. Then c^*c and hence c is bounded below.

DEFINITION 4. Let A be a von Neumann algebra acting on a Hilbert space H. The *relative α-topology* T_α on H is defined as the locally convex topology on H generated by the set of all seminorms p_M of the form $x\in H\to\sup\{|(x,y)|: y\in M_1\}$, where M varies F_α and F_α is the set all nonzero closed subspaces of H each of whose range projections belongs to A, having relative rank$<\alpha$. M_1 denotes the unit ball of M. When $A=B(H)$, the algebra of all bounded operators on H, we get the α-topology, by deleting "relative" in the above definition.

Throughout the paper, we fix a set S as the set of all T_α-neighborhoods s of 0 in H such that s is a finite intersection of sets $\{y\in H: p_M(y)<\varepsilon\}$, where M varies over F_α and ε varies over the set of all positive real numbers. Then S is a directed set with respect to the order $s\leq t$, meaning $t\subset s$. The next theorem is regarded as an analogue to various parts of ([5], [7], [8]).

THEOREM 1. *Let A be a semifinite von Neumann algebra and P the set of all projections in A. Assume that $\alpha \in \Gamma$. Then the following conditions are all equivalent.*

(i) $x \in J_\alpha$

(ii) *If $q \in P$, $q(H) \subset x(H)$, then rank$(q) < \alpha$ (extended Calkin condition.)*

(iii) *If $p \in P$ and x is bounded below on $p(H)$, then rank$(p) < \alpha$.*

(iv) *For every $\varepsilon > 0$, there is $p \in P$ such that $\|xp\| \leq \varepsilon$ and rank$(1-p) < \alpha$ (extended Rellich criterion.)*

(v) *$x|H_1 : H_1 \to H$ is continuous with respect to T_α on H and the norm on H where H_1 denotes the unit ball of H.*

(vi) *For every norm bounded net $\{\xi_s : s \in S\}$ in H such that $\xi_s \to 0$ in T_α, we have $x\xi_s \to 0$ in norm.*

Proof. (i) → (ii). Let $x \in J_\alpha$, and let $x \in P$, $q(H) \subset x(H)$. By Lemma 1, there is $y \in A$ such that $q = xy$. Since J_α is an ideal, $q \in J_\alpha$. Now, by the definition of J_α, we can find a sequence $q_n \in J_\alpha$ such that $\|q - q_n\| \to 0$. Then $\|q|q(H) - q_n|q(H)\| \to 0$. By Lemma 2, the operator $q_n|q(H) : q(H) \to H$ is bounded below for a suitably large n. Then $q_n q$ has the kernel $(I-q)(H)$ and has the closed range. On the other hand, $l(q_n q) \sim r(q_n q)$ in A, where $l(\cdot)$ and $r(\cdot)$ are the left and right supports respectively ([10]p. 94, Theorem 4.3). But $r(q_n q) = r(q)$. Consequently, $l(q_n q) \sim r(q)$ in A. It follows that rank(q) = rank $(q_n q) \leq$ rank $q_n < \alpha$, as desired.

(ii) → (iii) Let $p \in P$ and x be bounded below on $p(H)$. Then $r(xp) \sim l(xp)$. We put $l(xp) = q$. Thus, rank$(r(xp))$ = rank$(q) < \alpha$. Since ker(xp) = ker(p), we see that $r(xp) = p$, so rank $(p) < \alpha$.

(iii) → (iv) We quote Lemma 2.5([5] p.451) : Let $x \in A$, $\varepsilon > 0$ and $p_\varepsilon = E[0, \varepsilon)$, where E is the spectral measure of $|a|$. Then

(a) $\|a\xi\| < \varepsilon\|\xi\|$ for every $0 \neq \xi \in p_\varepsilon(H)$, once $p_\varepsilon(H) \neq (0)$,

(b) $\|a\xi\| \geq \varepsilon\|\xi\|$ for every $\xi \in (I - p_\varepsilon)(H)$,

(c) $n(a) \leq p_\varepsilon$, where $n(a)$ denotes the kernel projection for a.

(d) $p_\varepsilon|a| = |a|p_\varepsilon$.

Now we assume (iii). Let $\varepsilon > 0$. We put $p = p_\varepsilon$ in the quoted lemma above. Since x is bounded below on $(I-p)(H)$, we see that $I - p \in I_\alpha$ by (iii) and $\|xp\| \leq \varepsilon$ by (a) of the quoted lemma above.

(iv) → (i) Let $\varepsilon > 0$ be given arbitrarily. By (iv), there is $p \in P$ such that $\|xp\| \leq \varepsilon$ and $I - p \in I_\alpha$. Since $\|x - x(I-p)\| = \|xp\| \leq \varepsilon$, for an arbitrary $\varepsilon > 0$, while $x(I-p) \in I_\alpha$, we see that $x \in \bar{I}_\alpha = J_\alpha$, as desired.

(iv) → (v). Let $x \in J_\alpha$. We may assume that $x \neq 0$. Note that $x^* \in J_\alpha$, since J_α is a closed ideal in the C^*-algebra A. By the chain of implications (i) → (ii) → (iii) → (iv), which we already have shown, we conclude that (i) → (iv).

Thus, by the extended Rellich criterion, (iv), for every $\varepsilon>0$ there is an $M\in F_\alpha$ such that

$$\|x^*|M^\perp\|<\frac{\varepsilon}{4},\ \text{since } x^*\in J_\alpha.$$

Let $N=M^\perp$. Let $\eta_0\in H_1$ and

$$U_M(\eta_0)=\{\eta\in H:\frac{\sup}{\xi\in[x^*(M)]_1}|(\eta-\eta_0,\xi)|<\delta\},$$

noticing $\operatorname{rank}(x^*(I-p))\leq\operatorname{rank}(I-p)<\alpha$, where p is the projection in A whose range is N. Since the projection in A whose range is $\overline{x^*(M)}$ is exactly the range projection of $x^*(I-p)$, we see that $U_M(\eta_0)$ as defined above is an open neighborhood of η_0 with respect to T_α.

Now, for any $\eta\in U_M(\eta_0)\cap H_1$, we have

$$\begin{aligned}
\|x\eta-x\eta_0\|&=\sup_{\zeta\in H_1}|(x(\eta-\eta_0),\ \zeta)|\\
&\leq\sup_{\zeta\in M_1}|(x(\eta-\eta_0),\ \zeta)|+\sup_{\zeta\in N_1}|(x(\eta-\eta_0),\zeta)|\\
&=\sup_{\zeta\in M_1}|(\eta-\eta_0,\ x^*\zeta)|+\sup_{\zeta\in N_1}|(x(\eta-\eta_0),\ \zeta)|\\
&=\sup_{\zeta\in M_1}|(\eta-\eta_0,\ x^*\zeta)|+\sup_{\zeta\in N_1}|(\eta-\eta_0,\ x^*\zeta)|\\
&=\sup_{\zeta\in M_1}|(\eta-\eta_0,\ x^*\zeta)|+\|\eta-\eta_0\|(\varepsilon/4)\\
&=\sup_{\zeta\in M_1}\|x^*\||(\eta-\eta_0,\ (x^*\zeta)/\|x^*\|)|+2(\varepsilon/4)
\end{aligned}$$

(Note that $\|(x^*\zeta)/\|x^*\|\|\leq1.$)

$$\begin{aligned}
&\leq\sup_{\omega\in[x^*(M)]_1}\|x^*\|\|(\eta-\eta_0,\ \omega)|+(\varepsilon/2)\\
&<\|T\|\delta+\varepsilon/2.
\end{aligned}$$

Now we may assume that $\delta<\varepsilon/(2\|x\|)$. Then the last term above $<\varepsilon$, as desired.

(v) $\rightarrow$ (vi). This is clear.

(vi) $\rightarrow$ (v). Assume contrarily that $T|H_1$ were discontinuous at $\xi\in H_1$. Then for some open ball $B(x\xi,\varepsilon)$ centered at $x\xi$ with radius $\varepsilon>0$ in H, we have

$$x((s+\xi)\cap H_1)\not\subset B(x\xi,\varepsilon),\ \text{frequently for } s\in S.$$

Let $S_0=\{s\in S: x((s+\xi)\cap H_1)\not\subset B(x\xi,\ \varepsilon)\}$. For every $s\in S_0$, we find $\xi_s\in(s+\xi)\cap H_1$ and for every $s\in S\sim S_0$, we put $\eta_s=\xi$. Then $\{\eta_s: s\in S\}$ is a norm bounded net in H such that $\eta_s\rightarrow\xi$ in T_α. If we put $\xi_s=\eta_s-\xi$, then $\xi_s\rightarrow0$ in T_α while $\|x\xi_s\|=\|x\eta_s-x\xi\|\geq\varepsilon$, for all $s\in S_0$. That is, $\|x\xi_s\|\geq\varepsilon$ frequently for $s\in S$, contradicting to hypothesis (vi).

(v) $\rightarrow$ (iii). Let us assume that (v) is valid. Suppose that (iii) does not hold. Then there is $p\in P$ such that $\delta\|\xi\|\leq\|x\xi\|$ for all $\xi\in p(H)$ for some fixed $\delta>0$ and that rank $p\geq\alpha$. We put $K=p(H)$. We will construct a net $\{\xi_i: i\in I\}$ in K as follows. The index set I is just the family F_α. We can find a unit vector $\xi_i\in K$ such that x_i is orthogonal to $p(i)$.

Since $\aleph_0\leq\alpha$, we clearly see that $K\neq\{0\}$. Let $q\in A$ be the projection whose range is the closed subspace i of H. Let $q=\sum_{j\in J} q_j$, where each q_j is a nonzero finite projection in A and card$(J)<\alpha$. Clearly $p(i)\subset K$. We claim that $p(i)\subsetneq K$. Let $l(p(i))$ denote the projection whose range is $p(i)$. Thus $l(p(i))$ is just the left support $l(pq)$ of pq. Then $l(p(i))=p-p\wedge(I-q)$ ([10] p. 59, E. 2.3.). We claim that $p-p\wedge(I-q)\lneq p$. Assume contrarily that $p-p\wedge(I-q)=p$. Then $l(p(i))=p$, so rank $l(p(i))=$rank $p\geq\alpha$. On the other hand rank $l(p(i))=$rank $(p-p\wedge(I-q))=$rank$(q-(I-p)\wedge q)$ ([10] p. 94. Use Corollary 4.4 (ii), the parallelogram law: $p-p\wedge(I-q)\sim q-(I-p)\wedge q\leq$rank $q<\alpha$, recalling q is the projection whose range is i, while $i\in F_\alpha$. This is a contradiction. It follows that $l(p(i))\lneq p$. Hence we can find a unit vector $\xi_i\in K\theta p(i)$.

Now for an arbitary $M\in F_\alpha$, we put $i_0=M$. Then for every $i\geq i_0$ (meaning $i\supset i_0$), we see that

$$\begin{aligned}\sup_{\eta\in M_1}|(\xi_i,\ \eta)| &= \sup_{\eta\in M_1}|(p\xi_i,\ \eta)|\\ &= \sup_{\eta\in M_1}|(\xi_i,\ p\eta)|=0,\end{aligned}$$

since $\xi_i\perp p(i)$ while $p(i)\supset p(i_0)$ and $M_1\subset M=i_0$. It follows that $\xi_i\to 0$ in T_α. Since $\{\xi_i\}\subset H_1$, our hypothesis (v) implies that $\|x\xi_i\|\to 0$. Since $0<\delta\leq\|x\xi\|$, for all $\xi\in K_1$, while $\xi_i\in K_1$ we get the contradiction. We thus have shown that (v)$\to$(iii).

Reference

1. M. Breuer, *Fredholm theories in von Neumann algebras* I, Math. Ann. **178** (1968), 243-254.
2. ______ *Fredholm theories in von Neumann algebras* II, Math. Ann. **180** (1969), 313-325.
3. J. Dixmier, *Von Neumann algebras.* (English translation) North-Holland Publishing Co. 1981.
4. R. G. Douglas, *On majoriaztion factorization, and range inclusion of operators on Hilbert space,* Proc. Amer. Math. Soc. **17** (1966), 413-415.
5. V. Kaftal, *On the theory of compact operators in von Neumann algebras* I, Indiana Univ. Math. J. **26** (1977), 447-457.
6. ______ *On the theory of compact operators in von Neumann algebras* II, Indiana Univ. Math. J. **79** (1978), 129-137.
7. S.G. Lee, I.H. Lee, S.M. Kim and D.P. Chi, *A characterization of closed ideals in L(H)*, Proc. Coll. Natur. Sci., SNU. **8** (1983), 5-8.
8. S.G. Lee, *A general Calkin representation,* J. Korean Math. Soc. **20** (1983), 61-65.
9. M.G. Sonis, *On a class of operators in von Neumann algebras with Segal measure on the projectors,* Math. USSR Sbornik **13** (1971), 344-359.

10. S. Stratila and L. Zsido, *Lectures on von Neumann algebras*, Abacus Press, 1979.

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