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<**Original**>

Scaling Analysis of Core Flow Pattern in a Low-Aspect Ratio Rectangular Enclosure†

—(II) End-Driven Flow Regime—

Jinho Lee*

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종횡비가 낮은 직각형 밀폐용기 내의 코어흐름 형태에 관한 해석

—(II) 운동력이 양단에 존재하는 경우—

이 진 호

초 록

종횡비가 낮은 직사각형 밀폐용기 내에 Rayleigh 수가 충분히 커서 흐름의 운동력이 용기 양단에 존재하는 경우에 Part I 에서 개발된 해석적인 모델을 근거한 scaling analysis 를 통해 그 내부 흐름 형태를 정성적으로 예측, 기존의 결과와 비교, 검토하였다.

해석결과, Prandtl 수에 따라 여러가지 내부 흐름 형태가 존재할 수 있음이 밝혀졌으며 용기 내 뚜렷한 경계층 흐름이 존재하기 위한 필요조건도 아울러 얻어졌다.

1. Introduction

Due to stimuli from diverse applications, an ever increasing amount of research is being done on completely confined natural convection. Despite all the recent research activity, a central problem that has remained unsolved and is inherent to all confined convection situations is that the core flow pattern cannot be determined a priori from the given physical conditions. An attempt is, therefore, made herein to develop a method for obtaining a qualitative picture of the overall flow pattern from the given geometry, fluid, and boundary conditions. Because of the current interest in low-aspect ratio enclosures and because such configuration contains all the characteristics of the general problem, emphasis is being given to that configuration.

In Part I of this work consideration is given to low Rayleigh number situation in which the driving force exists in the core. An approximate criterion for the validity of analysis in that situation is given as $RaA^2 \leq 1$. Beyond this

† Presented at KSME Autumn Conference, 1983.
* Member, Department of Mechanical Engineering, Yonsei University

criterion, for a sufficiently high Rayleigh number (i,e., for $RaA^2 \gg 1$) there no longer exists the driving force in the core; instead, it comes from the end regions. Thus the flow would be of a boundary-layer type. Essentially, the difficulty existing in confined natural convection flow is associated with boundary-layer flow because of the intimate coupling between the boundary layer and core flows. The core flow is not readily determined from the boundary conditions but depends on the boundary layer, which, in turn, is influenced by the core. Since analytical boundary layer approach requires a priori knowledge of the core configuration at the outset, it is this coupling that constitutes the main source of difficulty in obtaining analytical solutions or even in getting qualitative ideas of the flow patterns to internal problems. This matter is not merely a subtlety for analysis but has equal significance for numerical and experimental studies.

Most of the existing work seems to ignore this point. In order to obtain solutions many authors either assume the core configuration, estimate it in an ad hoc manner, or use results obtained for similar problems. However, experience has shown that natural convection is very sensitive to the geometric configuration and boundary conditions so that utilization of results from "similar" problems is dangerous. It is also disturbing to note that velocities in natural-convection flows are normalized in many different ways even for indentical problems in the literature. This can lead to errors in analysis and considerable numerical problems.

In the present paper, by the same procedure as in Part I, consideration is given to the prediction of global core configurations at large Rayleigh numbers in which the flow is supposed to be driven by the end region. The proper conditions for relevant physical statements are explicitly delineated in the analysis and are compared with the available experimental observations. The results of Part I and the present work are combined and summarized in Table 1.

2. Global Core Configuration

For convenience, We rewrite the working form of the basic dimensionless equations (13) ~(15) in Part I. The analysis is made according to the Prandtl number, because physically for different Prandtl numbers different physical statements need to be made[1].

$$\frac{1}{\varepsilon_x}\frac{\partial(w,\psi)}{\partial(\eta,y)}+\frac{\partial(w,\psi)}{\partial(\zeta,y)}=\frac{\beta g\Delta T l^2 H}{\Psi^2{}_R}\left(\frac{1}{\varepsilon_x}\frac{\partial\theta}{\partial\eta}+\frac{\partial\theta}{\partial\zeta}\right)+\frac{\nu L}{\Psi_R H}\left(\frac{A^2}{\varepsilon^2{}_x}\frac{\partial^2 w}{\partial\eta^2}+2\frac{A^2}{\varepsilon^2{}_x}\frac{\partial^2 w}{\partial\eta\partial\zeta}+A^2\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{1}$$

$$w=-\frac{l^2}{H^2}\left(\frac{A^2}{\varepsilon_x{}^2}\frac{\partial^2\psi}{\partial\eta^2}+2\frac{A^2}{\varepsilon_x}\frac{\partial^2\psi}{\partial\eta\partial\zeta}+A^2\frac{\partial^2\psi}{\partial\zeta^2}+\frac{\partial^2\psi}{\partial y^2}\right) \tag{2}$$

$$\frac{1}{\varepsilon_x}\frac{\partial(\theta,\psi)}{\partial(\eta,y)}+\frac{\partial(\theta,\psi)}{\partial(\zeta,y)}=\frac{\alpha L}{\Psi_R H}\left(\frac{A^2}{\varepsilon^2{}_x}\frac{\partial^2\theta}{\partial\eta^2}+2\frac{A^2}{\varepsilon_x}\frac{\partial^2\theta}{\partial\eta\partial\zeta}+A^2\frac{\partial^2\theta}{\partial\zeta^2}+\frac{\partial^2\theta}{\partial y^2}\right) \tag{3}$$

$$\text{where } \varepsilon_x=\frac{\delta_x}{L} \tag{4}$$

2. 1. $Pr \gtrsim 1$(including $Pr \gg 1$)

(A) Core Flow Equations

As mentioned above, the heat transfer between the two end walls is dominated by convection. Heat transfer by conduction is thought to be important only in the boundary layers adjacent to the end walls. We thus balance the convection and conduction in the end region. In addition, since all the buoyancy force acts within the thermal boundary layers in the end region and the viscous effcous are important therein, for $Pr \gtrsim 1$, we can also balance the buoyancy and viscous forces in the end region, by means of

Table 1 Summary of the analysis

	Basic balances	Scales	Physical conditions			Core configurations
Core driven flow regime	Buoyancy～viscous in the core Horizontal viscous～vertical viscous in the end	$\psi_R \sim \frac{\beta g \Delta T H^4}{\nu L}$ $\delta_X \sim H$	$Gr\ A^2 \leqq 1$ $A^2 \ll 1$	$RaA^2 \ll 1$		Parallel flow pattern Linear temperature distribution
				$RaA^2 \sim 1$		Parreal flow patten Linear & stratified temperature distribution
	Buoyancy～inertia in the core Inertia in the end～inertia in the core	$\psi_R \sim (\beta g \Delta T H^3)^{1/2}$ $\delta_X \sim L$	$Gr_H{}^2 \gg 1$ $A^2 \ll 1$	$PrRaA^2 \ll 1$		Non-paralle flow pattern Linear temperature distribution
Boundary-layer flow regime	Convection～conduction in the end Buoyancy～viscous in the end	$\psi_R \sim \alpha Ra^{1/4}$ $\delta_X \sim \frac{H}{Ra^{1/4}}$	$RaA^2 \gg 1$ $A^2 \ll 1$	$ARa^{1/4} > 1$	$Pr \sim 1$	Distinct horizontal thermal layers exist. stratified temperature distribution with stagnat fluid motin in the mid-core.
					$Pr \gg 1$	Distinct horizontal thermal layers exist. parallel flow pattern stratified temperature distribution in the mid-core.
	Conv.～Cond. in the end Buoy.～Inertia in the end	$\psi_R \sim \alpha \cdot (PrRa)^{1/4}$ $\delta_X \sim \frac{H}{(PrRa)^{1/4}}$		$A(PrRa)^{1/4} > 1$	$Pr < 1$*	Distinct horizontal layers exist. stratified temperature distribution with stagant fluid motion in the mid-core

* This always includes the case of $Pr \ll 1$

which the stretching parameter, ε_x, can also be determined.

From Eq. (3) the balance between convection and conduction in the end regions can be represented as

$$\frac{1}{\varepsilon_x} \sim \frac{\alpha L}{\Psi_R H} \frac{A^2}{\varepsilon_x{}^2} \tag{5}$$

From Eq. (1), the balance between buoyancy and viscous forces in the end regions can be represented as

$$\frac{\beta g \Delta T l^2 H}{\Psi_R{}^2} \frac{1}{\varepsilon_x} \sim \frac{\nu L}{\Psi_R H} \frac{A^2}{\varepsilon_x{}^2} \tag{6}$$

Considering that the flow is driven by the buoyancy force in the end regions, the vorticity would be dominant therein. It is thus appropriate to represent the characteristic vorticity, Ω_R, by specifying the characteristic length l by δ_x as

$$l = \delta_x \tag{7}$$

and

$$\Omega_R = \frac{\Psi_R}{\delta_x{}^2} \tag{8}$$

From (5)～(7), we then obtain

$$\Psi_R \sim Ra^{1/4} \tag{9}$$

and

$$\varepsilon_x \sim \frac{A}{Ra^{1/4}} \tag{10}$$

From (4) and (10) we find

$$\delta_x \sim \frac{H}{Ra^{1/4}} \tag{11}$$

In (11), it is seen that the end region characteristic length scale δ_x is similar to the familiar boundary thickness of a vertical flate for high Pr[2].

Substituting Ψ_R, ε_x and δ_x into Eqs. (1)～(3) and considering the derivatives with respect to

ζ and y, the equations which will describe the core flow can be written as.

$$\frac{1}{Pr}\ \frac{\partial(w,\psi)}{\partial(\zeta,y)}=\frac{\partial\theta}{\partial\zeta}+\frac{1}{ARa^{1/4}}\left(A^2\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{12}$$

$$w=-\frac{1}{Ra^{1/4}}\left(A^2\frac{\partial^2\psi}{\partial\zeta^2}+\frac{\partial^2\psi}{\partial y^2}\right) \tag{13}$$

$$\frac{\partial(\theta,\psi)}{\partial(\zeta,y)}=\frac{1}{ARa^{1/4}}\left(A^2\frac{\partial^2\theta}{\partial\zeta^2}+\frac{\partial^2\theta}{\partial y^2}\right) \tag{14}$$

Let us look at the core flow characteristics based on the dimensionless parameter in the Eqs. (12)~(14).

(B) Core flow characteristics

(1) $\frac{1}{ARa^{1/4}}<1,\ A^2\ll 1$

In Eq. (12), when $\frac{1}{ARa^{1/4}}<1$ (or $ARa^{1/4}>1$) the conduction terms are negligible compared to the convection terms. But as these terms are the derivatives of the highest order in the equation, for negligible horizontal conduction term the equation is singular with respect to y in the core. This implies that there exists a thin layer(it is called here "horizontal thermal layer") adjacent to the horizontal boundaries within which the vertical conduction term becomes important. This horizontal layer can be estimated by the usual coordinate stretching method. For this we introduce the transformation as

$$y=\varepsilon_y\tilde{y} \tag{15}$$

where

$$\varepsilon_y=\frac{\delta_y}{H} \tag{16}$$

and δ_y represent the horizontal thermal layer.

Substituting the derivatives of (15) into Eq. (12) and balancing the convection and conduction within δ_y, we then have

$$\frac{1}{\varepsilon_y}\sim\frac{1}{ARa^{1/4}}\ \frac{1}{\varepsilon_y^{\,2}} \tag{17}$$

From (16) and (17), We find

$$\delta_y\sim\frac{L}{Ra^{1/4}} \tag{18}$$

or

$$\frac{\delta_y}{H}\sim\frac{1}{ARa^{1/4}} \tag{19}$$

In order that the horizontal thermal layer be distinct.

$$\frac{\delta_y}{H}<1 \tag{20}$$

so that

$$ARa^{1/4}>1 \tag{21}$$

Under the condition (21), to a good approximation, the core flow equations (12) and (14) reduce to

$$\frac{1}{Pr}\ \frac{\partial(w,\psi)}{\partial(\zeta,y)}=\frac{\partial\theta}{\partial\zeta} \tag{22}$$

$$\frac{\partial(\theta,\psi)}{\partial(\zeta,y)}=0 \tag{23}$$

Eqs. (22) and (23) will thus describe core configuration outside the horizontal thermal layer, i. e., in the mid-core region, under the condition (21). Since Pr acts as a parameter in Eq. (22), we first examine the core flow characteristics for high Pr ($Pr\gg 1$) and then for moderate Pr ($Pr\sim 1$).

(i) $Pr\gg 1$

For $Pr\gg 1$, the inertia term in Eq. (22) becomes negligible and we have

$$\frac{\partial\theta}{\partial\zeta}=0 \tag{24}$$

From (24), two possible temperature profiles can be obtained as

$$\theta_c=\theta_c(y) \tag{25}$$

or

$$\theta_c=\text{const.} \tag{26}$$

where the subscript c is affixed to denote the temperature distribution outside the horizontal thermal layer. Among the two possible core configurations, it can be shown that the case θ_c=const is not a possible temperature profile to the situation concerned herein[3]. From (23) and (25), we thus find

$$\frac{\partial\psi}{\partial\zeta}=0 \tag{27}$$

and

$$\psi_c = \psi_c(y) \tag{28}$$

From (27) and (28), it therefore can be seen that for $Pr \gg 1$, under the condition $ARa^{1/4} > 1$ there exists a horizontal thermal layer in the core and the temperature distribution outside the horizontal thermal layer, i. e., in the mid-core, is stratified while the corresponding core flow structure is parallel. This parallel core flow structure was observed in the experiment of Ostrach et.al.[4] for $Pr = 1.38 \times 10^3$, $A = 0.1$ and $Ra \sim 10^6$, for the semi-conducting horizontal walls.

(ii) $Pr \sim 1$,

When $Pr \sim 1$, (in fact Pr is about 1 or slightly greater than 1), the inertia term in Eq. (22) is not negligible. Instead the circulating flow will transport vorticity across the cavity by the inertia and, as can be conceivable by the singular behaviour in the equation, the diffusion of vorticity will be important within a layer (we call it here "horizontal viscous layer" in distinction to the horizontal thermal layer) along the horizontal boundarise. This horizontal viscous layer can be estimated from the balance between the vorticity transport and diffusion terms within that layer. But in this case, we have to use a modified characteristic stream function $\tilde{\Psi}_R$ instead of Ψ_R, because the characteristic stream function has different values according to the different characteristic length scales within which different balances are made. In Appendix the modified characteristic stream function $\tilde{\Psi}_R$ is estimated. The argument of the necessity of the distinction between the two characteristic stream function is also given therein.

For an estimate of the horizontal viscous layer, we introduce the transformation as

$$y = \varepsilon_v \tilde{y} \tag{29}$$

where

$$\varepsilon_v = \frac{\delta_v}{H} \tag{30}$$

and δ_v is the horizontal viscous layer.

Substituting the derivatives of (29) into Eq. (12) with the modified characteristic stream function $\tilde{\Psi}_R$ replacing Ψ_R, from the balance between the vorticity transporty and diffusion terms within δ_v, we have

$$\frac{1}{\varepsilon_v} \sim \frac{\nu L}{\tilde{\Psi}_R H} \frac{1}{\varepsilon_v^2} \tag{31}$$

From Appendix, as

$$\tilde{\Psi}_R \sim \alpha Pr^{1/2} Ra^{1/4} \tag{32}$$

from (30)~(32), we find

$$\delta_v \sim \frac{Pr^{1/2} L}{Ra^{1/4}} \tag{33}$$

or

$$\frac{\delta_v}{H} \sim \frac{Pr^{1/2}}{ARa^{1/4}} \tag{34}$$

For distinct horizontal viscous layer,

$$\frac{\delta_v}{H} < 1 \tag{35}$$

so that

$$ARa^{1/4} > Pr^{1/2} \tag{36}$$

Since $Pr \sim 1$, from (18) and (33)

$$\frac{\delta_v}{\delta y} \sim Pr^{1/2} \sim 1 \tag{37}$$

Under the condition (36), the effect of viscous shear in the core is confined to the horizontal viscous layer, δ_v. Here the flow is driven by the boundary layer in the end region and there is no other way to induce any fluid motion in the core except by the viscous shear of the end driven cirulatiog flows. Thus, as $\delta_v \sim \delta_y$ from (37), most of the flow in the core will thus circulate through horizontal viscous layer adjacent to the horizontal boundaries. Outside that layer, the flow which may result from the entrainment-detrainment of the end driven core circulating flow would be of much lower than the circulating flow so that the motion therein would be almost stagnant. Outside the horizontal viscous layer, we may thus put

$$\psi_c = \text{const.} \tag{38}$$

Then since $\delta_v \sim \delta_y$, from Eq (22) we find

$$\frac{\partial \theta}{\partial \zeta} = 0 \tag{39}$$

and this gives

$$\theta_c = \theta_c(y) \tag{40}$$

as mentioned previously. For $\psi_c = \text{const.}$, there is no convection at all outside δ_y and the energy equation (23) is automatically satisfied.

From (38) and (40), it is thus seen that for $Pr \sim 1$, under the condition $ARa^{1/4} > 1$ there exists the horizontal thermal layer in the core and the temperature distribution in the mid-core will be stratified while the fluid motion therein would be almost stagnant. This stagnant core configuration was observed in the experiment of Al-Homoud[5] for $Pr = 7.0$, $A = 0.0625$ and $Ra = 2.0 \times 10^8 \sim 2.0 \times 10^9$. This shows good agreement with the above prediction.

(2) $\frac{1}{ARa^{1/4}} > 1$, $A^2 \ll 1$

Under this condition, the conduction term becomes important in Eq (14) and from (19), the horizontal thermal layer δ_y becomes of order

$$\frac{\delta_y}{H} \sim \frac{1}{ARa^{1/4}} > 1 \tag{41}$$

There thus will no longer exist distinct horizontal thermal layers in the core. Instead some horizontal temperature gradient will exist in the core which may develop fluid motion in the core in addition to the end driven circulating flow. Further it is supposed that the thermal boundary layer structure in the end region may somewhat be altered due to the core temperture gradient. Thus the resulting flow driving mechanism would be modified from the strict end region boundary-layer driven flow mechanism. The flow characteristics in this situation are, therefor, supposed to lie between those in the core-driven flow regime and in the strict boundary-layer driven flow regime. In this sense this flow regime may be called the "Intermediate Flow Regime". Global core flow characteristics in the intermediate flow regime need a spearate consideration, because it is not clear whether the characteristic length scales would be geometric or not.

From the analysis, it can be seen that the condition, $ARa^{1/4} > 1$, in (21) is a necessary condition for existence of a distinct boundary-layer flow regime for $Pr \gtrsim 1$ (including $Pr \gg 1$). This condition agrees well with the available experimental data[5~7]. Recent numerical works [8~10] identified the existence of horizontal boundary layers and their results also show good agreement with the above prediction. Detailed core velocity and temperature profiles in the boundary-layer flow regime are given by Tichy and Gadgil[11]

2. 2. $Pr < 1$*

(A) Core Flow Equations

For $Pr < 1$, the flow boundary-layer extends less than the thermal boundary layer and the main body of fluid can be considered to be inviscid within the thermal boundary layer except in the vicinity of end walls. Since all the buoyancy force acts within the thermal boundary-layer, in addition to the balance between convection and conduction in the end regions, we balance the buoyancy and inertia forces in the end regions from which the stretching parameter, ε_x, can be determined.

From Eq. (1), the balance between buoyancy and intertia forces in end region can be represented as

$$\frac{1}{\varepsilon_x} \sim \frac{\beta g \Delta T l^2 H}{\Psi_R^2} \frac{1}{\varepsilon_x} \tag{42}$$

Then, with the balance between convection and condution in the end region in (5), from(7) and (42) we obtain

$$\Psi_R \sim \alpha (PrRa)^{1/4} \tag{43}$$

and

* This always includes the case of $Pr \ll 1$

$$\varepsilon_x \sim \frac{A}{(PrRa)^{1/4}} \tag{44}$$

From (4) and (44), We find

$$\delta_x \sim \frac{H}{(PrRa)^{1/4}} \tag{45}$$

which is similar to the familiar boundary-layer thickness of a vertical flat plate for low Pr[2]

Substituting Ψ_R, ε_X and δ_x into Eqs. (1)~(3) and considering the derivatives with respect to ζ and y, the core flow equations can be represented as

$$\frac{\partial(w,\phi)}{\partial(\zeta,y)}=\frac{\partial\theta}{\partial\zeta}+\frac{Pr}{A(PrRa)^{1/4}}\left(A^2\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{46}$$

$$w=-\frac{1}{(PrRa)^{1/2}}\left(A^2\frac{\partial^2\phi}{\partial\zeta^2}+\frac{\partial^2\phi}{\partial y^2}\right) \tag{47}$$

$$\frac{\partial(\theta,\phi)}{\partial(\zeta,y)}=\frac{1}{A(PrRa^{1/4})}\left(A^2\frac{\partial^2\theta}{\partial\zeta^2}+\frac{\partial^2\theta}{\partial y^2}\right) \tag{48}$$

(B) Core Flow Characteristics

(1) $\frac{1}{A(PrRa)^{1/4}}<1,\ A^2\ll 1$

When $\frac{1}{A(PrRa)^{1/4}}<1$, both the viscous and heat diffusion terms become negligible in Eqs. (46) and (48) and as in the previous analysis, the horizontal layer can be estimated by the coordinate stretching method.

Substituting the derivatives of (15) into Eq. (48) and balancing the convection and conduction within δ_y, we have

$$\frac{1}{\varepsilon_y}\sim\frac{1}{A(PrRa)^{1/4}}\ \frac{1}{\varepsilon_y^{\,2}} \tag{49}$$

From (16) and (49), we then find

$$\frac{\delta_y}{H}\sim\frac{1}{A(PrRa)^{1/4}} \tag{50}$$

In order that the horizontal thermal layer be distinct.

$$\frac{\delta_y}{H}<1 \tag{51}$$

or

$$A(PrRa)^{1/4}>1 \tag{52}$$

Under the condition (52), to a good approximation, outside the horizontal layer δ_y, i.e., in the mid-core, the core flow equations (46) and (48) reduce to

$$\frac{\partial(w,\phi)}{\partial(\zeta,y)}=\frac{\partial\theta}{\partial\zeta} \tag{53}$$

$$\frac{\partial(\theta,\phi)}{\partial(\zeta,y)}=0 \tag{54}$$

For $Pr<1$, as can be seen from the works of Ostrach[12] and Sparrrow and Gregg[13], the thickness of velocity boundary layers along the end walls is less than (or about equal to) that of thermal boundary layers. In the core, it is also expected that the horizontal viscous layer is less than (or about equal to) the horizontal thermal layer. In this situation, as mentioned in Appendix, the value of characteristic stream function with in the horizontal thermal layer will be identical to that within the horizontal viscous layer. There thus is no need to modify the characteristic stream function for the estimation of horizontal viscous layer. Then, by introducing the transformation of (29) into Eq. (46) and from the balance between the vorticity transport and diffusion terms within the horizontal viscous layer δ_V, we find

$$\frac{\delta_V}{H}\sim\frac{Pr}{A(PrRa)^{1/4}} \tag{55}$$

For a distinct horizontal viscous layer

$$\frac{\delta_V}{H}<1 \tag{56}$$

so that

$$A(PrRa)^{1/4}>Pr \tag{57}$$

Under the condition (57), the effect of viscous shear in the core will be confined to the horizontal viscous layer and since $\delta_V<\delta_Y$ from (50) and (55) for low Pr, most of the flow in the core will circulate through the horizontal thermal layer. Then the fluid motion outside the horizontal thermal layer, as mentioned earlier in the previous analysis, would remain nearly stagnant. We may thus put

$$\psi_c = \text{const.} \tag{58}$$

in the mid-core region.

Then, from Eq. (53) we find

$$\frac{\partial \theta}{\partial \zeta} = 0 \tag{59}$$

and thus,

$$\theta_c = \theta_c(y) \tag{60}$$

It can be seen from (58) and (60) that for $Pr<1$, under the condition $A(PrRa)^{1/4}>1$, with the the distinct horizontal theremal layer in the core, the temperature distribution will be stratified in the mid-core while the fluid motion therein would remain almost stagnant. Unfortunately no experiments have, as yet, been available for $Pr<1$ to verify the above prediction.

(2) $\frac{1}{A(PrRa)^{1/4}}>1,\ A^2\ll 1$

Under this condition, from Eq. (48), conduction becomes important in the core and from Eq. (50), the horizontal thermal layer, δ_y, becomes of order

$$\frac{\delta_y}{H} \sim \frac{1}{A(PrRa)^{1/4}} > 1 \tag{61}$$

Thus, there will no longer exist the distinct horizontal thermal layer in the core and instead, some core fluid motion can be induced by the horizontal temperature gradients in the core in addition to the boundary-layer driven circulating flows. As a consequence, the theremal boundary layer structure in the end region and thus the resulting flow driving mechanism will be modified according to the core temperature gradients. This flow regime also corresponds to the "intermediate flow regime" mentioned previously. The condition (52), i.e., $A(PrRa)^{1/4}>1$, should thus be the necessary condition for the distinct boundary-layer flow regime. Experiment for $Pr<1$ is needed for the verification of the above prediction.

3. Summary and Concluding Remarks

Consideration has been given to the prediction of global core configurations at large Rayleigh numbers in a low aspect-ratio rectangular enclosure.

In the analysis, the balance was made between convection and conduction in the end region. In addition, the balance was made between buoyancy and viscous forces for $Pr\gtrsim 1$ (including $Pr\gg 1$) and between buoyancy and inertia forces for $Pr<1$ (this always includes the case of $Pr\ll 1$). Under the conditions, $ARa^{1/4}>1$ for $Pr\gtrsim 1$ (including $Pr\gg 1$) and $A(PrRa)^{1/4}>1$ for $Pr<1$, there exist distinct horizontal thermal layers adjacent to the horizontal boundaries in the core and the temperature distribution outside the horizontal thermal layers, i.e., in the mid-core, is stratified. The core flow pattern is parallel for $Pr\gg 1$ while the fluid motion remains nearly stagnant in the mid-core for $Pr\sim 1$ and $Pr<1$. For $RaA^2\gg 1$, but when $ARa^{1/4}<1$ for $Pr\gtrsim 1$ (including $Pr\gg 1$) and when $A(PrRa)^{1/4}<1$ for $Pr<1$, the flow regime correspond to the intermediate flow regime and needs a sparation consideration. Summary of the results of Part I and the present analysis is given in Table 1.

By comparisons, predictions of core configurations made herein show good agreement with the existing experimental and numerical results for $Pr\sim 1$ and $Pr\gg 1$, although the data are not extensive. More experiments (especially for low Pr) are needed to verify the present predictions in order to understand the physics of the core flow pattern more clearly and to indicate the validity of the type of analysis presented herein.

Based on the multiple scales techniques, global prediction of the core flow pattern is generally satisfactory. It is thought that employing the ideas of multiple scales technique attempted herein can surely be applied to resolve many other complex problems.

References

(1) Ostrach, S., "Laminar Flows with Body Forces", in High Spede Aerodynamics and Jet Propulsion (F.K. Moore, edit), Vol. 4, Chap. F, Princeton Univ. Press, Princeton, New Jersy, 1964

(2) Jaluria, Y., "Natural Convection Heat and Mass Transfer", Chap. 3, Pergamon Press, New York, 1980

(3) Lee, J. and Ostrach, S., "Prediction of Natural Convection Flow Pattern in Low-Aspect Ratio Enclosures", FTAS/TR-82-158, Case Western Reserve Univ., Cleveland, Ohio, 1982

(4) Ostrach, Loka, R.R. and Kumar, A., "Natural Convection in Enclosures", edit. by Torrance and Catton, ASME-Vol. 18, 1980

(5) Al-Homoud, A.A., "Experimental Study of High Rayleigh Number Convection in Horizontal Cavity with Different End Temperatures", M.S. Thesis, Univ. of Colorado, Boulder, Colorado, 1979

(6) Imberger, J., "Natural Convection in a shallow Cavity with Differentially Heated End Walls. Part 3. Experimental Results", J. Fluid Mech., Vol. 65, pp.247~260, 1974

(7) Kamotani, Y., Wang, L.W. and Ostrach, S., "Experiments on Natural Convection Heat Transfer in Low Aspect Ratio Enclosures", AIAA Journal, Vol. 21, pp.290~294, 1983

(8) Shiralkar, G.S. and Tien, C.L., "A Numerical Study of Laminar Natural Convection in Shallow Cavities", J. Heat Transfer, Vol. 103, pp.226-231, 1981

(9) LEE, E.I. and Sernas, V., "Numerical Study of Heat Transfer in Rectangular Euclosures of Aspect Ratio Less Than One", ASME Paper 80-WA/HT-43, 1980

(10) Shiralkar, G., Gadgil, A. and Tien, C.L., "High Rayleigh Number Convection in Shallow Enclosures with Different End Temperatures", Int. J. Heat Mass Transfer, Vol. 24, pp.1621~1629, 1981

(11) Tichy, J. and Gadgil, A., "High Rayleigh Number Laminar Convection in Low Aspect Ratio Enclosures with Adiabatic Horizontal Walls and Differentially Heated Vertical Walls", J. Heat Transfer, Vol. 104, pp.103~110, 1982

(12) Ostrach, S., "An Analysis of Laminar Free-Convection Flow and Heat Transfer About a Flat Plate Parallel to the Direction of the Generating Body Force", NACA Rep. 1111, 1953

(13) Sparrow, E.M. and Gregg, J.L., "Details of Exact Low Prandtl Number Boundary-Layer Solutions for Forced and for Free Convection", NASA Memorandum 2-27-59E, 1958

Appendix

Modified Characteristic Stream Function $\tilde{\Psi}_R$

In the boundary layer equation, in normalizing the velocity components within the thermal or flow boundary layers we use the same characteristic velocity, U_R, because the characteristic velocity is of the same order of magnitude within both boundary layers. On the contrary, in vorticity transport equation of the boundary layer, since the characteristic stream function in represented as

$$\Psi_R \sim l U_R \qquad (A.1)$$

where l is the characteristic length, even for the same value of U_R, the characteristic stream fuuction changes according to the characteristic length scale, l, and thus one should be careful in normalizing the vorticity equation, Actually, in the case of large Prandtl numbers, the flow boundary layer extends beyond the thermal boundary layer, thus characteristic stream function differs within each layer and it needs to be determined according to the corresponding boundary layers.

For low Prandtl numbers, however, although the thermal boundary layer extends beyond the flow boundary layer, since the main body of fluid flows within the flow boundary layer, the characteristic stream function will be of the same order within both boundary layers. In Eq. (9), the estimate of the characteristic stream function, Ψ_R, was made for the thermal boundary layer, say δ_t, for $Pr \gtrsim 1$ (including $Pr \gg 1$). Based on the above argument, for the same Prandtl numbers, we now determine the charateristic stream function, $\tilde{\Psi}_R$, for the flow boundary layer.

From (A.1), U_R can bewritten as

$$U_R \sim \frac{\Psi_R}{\delta_t} = \frac{\Psi_R}{\delta_f} \frac{\delta_f}{\delta_t} = \frac{\tilde{\Psi}_R}{\delta_f} \qquad (A.2)$$

where $\tilde{\Psi}_R = \frac{\delta_f}{\delta_t} \Psi_R$, is the characteristic stream function modified for the flow boundary layer, δ_f. To estimate $\tilde{\Psi}_R$, we replace Ψ_R by $\tilde{\Psi}_R$ in Eq. (1) and introduce the multiple scales for the flow boundary layer, δ_f, as

$$\zeta = x, \eta = \frac{x}{\bar{\varepsilon}_x} \qquad (A.3)$$

where

$$\bar{\varepsilon}_x = \frac{\delta_f}{L} \qquad (A.4)$$

Then substituting the derivatives of (A.3) into Eq. (1) and balancing the intertia and viscous forces within δ_f in the end, we have

$$\frac{1}{\bar{\varepsilon}_x} \sim \frac{\nu L}{\Psi_R H} \frac{A^2}{\bar{\varepsilon}_x^2} \qquad (A.5)$$

and from(A.2) and (A.5), we find

$$\bar{\varepsilon}_x \sim \frac{\nu A}{\tilde{\Psi}_R} \sim \frac{\nu A}{\Psi_R} \frac{\delta_t}{\delta_f} \qquad (A.6)$$

For δ_t, since from(5)

$$\varepsilon_x \sim \frac{\alpha A}{\Psi_R} \qquad (A.7)$$

From (A.6) and (A.7), we have

$$\bar{\varepsilon}_x \sim Pr \frac{\delta_t}{\delta_f} \varepsilon_x \qquad (A.8)$$

or

$$\frac{\bar{\varepsilon}_x}{\varepsilon_x} \sim Pr \frac{\delta_t}{\delta_f} \qquad (A.9)$$

Then, from (4), (A.4) and (A.9)

$$\frac{\bar{\varepsilon}_x}{\varepsilon_x} \sim \frac{\delta_f}{\delta_t} \sim Pr \frac{\delta_t}{\delta_f} \qquad (A.10)$$

and thus,

$$\frac{\delta_f}{\delta_t} \sim Pr^{1/2} \qquad (A.11)$$

Since from(11)

$$\delta_t \sim \frac{H}{Ra^{1/4}} \qquad (A.12)$$

From (A.11) and (A.12), we find

$$\delta_f \sim \frac{Pr^{1/2}}{Ra^{1/4}} H \qquad (A.13)$$

and thus,

$$\tilde{\Psi}_R = \frac{\delta_f}{\delta_t} \Psi_R \sim \alpha Pr^{1/2} Ra^{1/4} \qquad (A.14)$$

witg this modified characteristic stream function, $\tilde{\Psi}_R$, in (A.14), the estimate of the horizontal viscous layer was made in (33).