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On Testing Multisample Sphericity in the Complex Case[+]

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ABSTRACT

In this paper, likelihood-ratio test has been derived for testing multisample sphericity in complex multivariate Gaussian populations. The h^{th} moment of the test statistic is given and its exact distribution has been derived using inverse Mellin transform. Asymptotic distribution of the test statistic is also given.

1. Introduction

Let $\underset{\sim}{Z}'=(\underset{\sim}{Z}_1', \cdots, \underset{\sim}{Z}_q')$ be distributed as complex multivariate normal (see Goodman, 1963) with mean vector $\underset{\sim}{\mu}'=(\underset{\sim}{\mu}_1', \cdots, \underset{\sim}{\mu}_q')$ and Hermitian positive definite covariance matrix $\Sigma=(\Sigma_{ij})$ with $E\{(\underset{\sim}{Z}_i-\underset{\sim}{\mu}_i)(\overline{\underset{\sim}{Z}_j-\underset{\sim}{\mu}_j})'\}=\Sigma_{ij},\ i,j=1, \cdots, q$. Let each $\underset{\sim}{Z}_i$ be of order $p\times 1$ and $\Sigma_{ij}=0,\ i\neq j=1, \cdots, q$. Under this set-up consider testing the following hypothesis.

$$H: \Sigma_{11}=\Sigma_{22}=\cdots \Sigma_{qq}=\sigma^2 I_p \tag{1.1}$$

against general alternatives H_1 : not H, where $\sigma^2>0$ is unknown and I_p is the identity matrix of order p.

It may be pointed out that complex multivariate Gaussian distribution has been found very useful in such areas as physics and time series analysis(see Goodman, 1963; Gupta, 1973). Bronk (1965) has shown that under certain conditions the distribution of the energy levels of atomic nuclei is the distribution of the roots of a complex random

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matrix. Brillinger (1969) has shown that the asymptotic distribution of the matrix of sepctral densities of a strictly stationary time series is complex Wishart.

In this article we propose the likelihood ratio test for testing H and derive its distribution (see Gupta 1971, 1973, 1976). This problem in the real case has already been studied by Mandoza (1980), and Gupta and Nagar (1984). The moments of the likelihood ratio statistic are derived in Section 2. Exact distributions in terms of G-function and in a series form are given in Sections 3 and4 respectively. Asymptotic distribution is given Section 5.

2. Likelihood Ratio Test Statistic and its Moments

By using the definition of likelihood ratio test statistic it is easy to see that the test statistic Λ for testing H is

$$\Lambda=\frac{N_0^{N_0 p}}{\prod_{i=1}^{q} N_i^{N_i p}}\frac{\prod_{i=1}^{q}|A_{ii}|^{N_i}}{(\mathrm{tr}A/p)^{N_0 p}},$$

where

$$A_{ii}=\sum_{j=1}^{N_i}(\underset{\sim}{Z}_{ij}-\underset{\sim}{Z}_{i.})\overline{(\underset{\sim}{Z}_{ij}-\underset{\sim}{Z}_{i.})}',\quad A=A_{11}+A_{22}+\cdots+A_{qq},$$

$$\underset{\sim}{Z}_{i.}=\sum_{j=1}^{N_i}\underset{\sim}{Z}_{ij}/N_i,\quad N_0=N_1+N_2+\cdots+N_q,$$

and $\underset{\sim}{Z}_{ij}$ is the j-th $(j=1,\cdots,N_i)$ independent observation on $\underset{\sim}{Z}_i$ $(i=1,\cdots,q)$. Let $n_i=N_i-1$ and $n_0=N_0-q$, then the modified likelihood ratio test statistic, is

$$\Lambda^*=\frac{n_0^{n_0 p}}{\prod_{i=1}^{q} n_i^{n_i p}}\frac{\prod_{i=1}^{q}|A_{ii}|^{n_i}}{(\mathrm{tr}A/p)^{n_0 p}}. \tag{2.1}$$

By arguments similar to the real case (see Mandoza, 1980), the h-th moment of Λ^* is

$$E(\Lambda^{*h})=\frac{(n_0 p)^{n_0 ph}\,\Gamma[n_0 p]}{\prod_{i=1}^{q} n_i^{phn_i}\,\Gamma[p(n_0+n_0 h)]}\prod_{j=1}^{p}\prod_{i=1}^{q}\frac{\Gamma[n_i(1+h)+1-j]}{\Gamma[n_i+1-j]}. \tag{2.2}$$

When the sample sizes are equal $(N_i=N)$, the h-th moment of the statistic $\Lambda^{*1/n}=\Lambda^{1/N}=V$ is

$$E(V^h)=(pq)^{pqh}\frac{\Gamma[npq]}{\Gamma[pq(n+h)]}\cdot\prod_{j=1}^{p}\frac{\Gamma^q[n+h+1-j]}{\Gamma^q[n+1-j]}. \tag{2.3}$$

It may be noted that for $q=1$, the hypothesis H defined in (1.1) is the usual

Mauchly's (1940) sphericity hypothesis and the distribution of V in this case is the distribution of the likelihood ratio statistic for testing sphericity of a complex Gaussian model (see Gupta, 1977). For $p=1$, Gupta and Tang (1983) have derived the exact distribution and have also tabulated the percentage points.

3. Distribution of Likelihood Ratio Test Statistic

By using inverse Mellin transform and the expression (2.3), the probability density function of V is given by

$$(3.1)\qquad f(v)=(2\pi\omega)^{-1}\int_C (pq)^{pqh}\frac{\Gamma\, npq}{\Gamma[pq(n+h)]}\prod_{j=1}^{q}\frac{\Gamma^q[n+h+1-j]}{\Gamma^q[n+1-j]}v^{-1-h}\,dh \quad 0<v<1,$$

where $\omega=(-1)^{1/2}$ and C is a contour selected suitably.

Simplifying the integrand in the above expression by applying Gauss multiplication formula (Luke, 1969, p. 11) to $\Gamma[pq(n+h)]$, writing $\Gamma[n+h+1-1]\cdots\Gamma[n+h+1-p]$ in the reverse order as $\Gamma[n+h-p+1]\ \Gamma[n+h-p+2]\cdots\Gamma[n+h-p+p]$, and substituting $\alpha=n+h-p$ in the integrand, one obtains

$$(3.2)\qquad f(v)=K(n,\ p,\ q)(2\,\pi\omega)^{-1}\,v^{n-p-1}\int_{C_1}\frac{\prod_{j=1}^{p}\Gamma^q[\alpha+j]}{\prod_{j=0}^{pq-1}\Gamma\left[\alpha+p+\frac{j}{pq}\right]}v^{-\alpha}\,d\alpha,\quad 0<v<1,\quad \text{where } C_1$$

is the changed contour, and

$$(3.3)\qquad K(n,\ p,\ q)=(2\,\pi)^{\frac{pq-1}{2}}\Gamma[npq]/\{(pq)^{npq-1/2}\prod_{j=1}^{p}\Gamma^q[n+1-j]\}\,.$$

From the definition of Meijer's G-function (Luke, 1969, p. 143), it is easy to see that (3.2) can be written as

$$(3.4)\qquad \begin{aligned} f(v)&=K(n,\ p,\ q)\,v^{n-p-1}\,G^{pq,0}_{pq,pq}\left[v\,\middle|\,\begin{matrix}\left\{p+\frac{j}{pq}\right\},\ j=0,\ \cdots,\ pq-1\\ \{j,\ \text{repeated } q \text{ times}\},\ j=1,\ \cdots,\ p\end{matrix}\right]\\ &=K(n,\ p,\ q)\,v^{n-p-1}\,G^{pq-1,0}_{pq-1,pq-1}\left[v\,\middle|\,\begin{matrix}\left\{p+\frac{j}{pq}\right\},\ j=1,\cdots2,\ ,pq-1\\ \{j,\ \text{repeated } q \text{ times}\},\ j=1,\cdots,p-1,\ \{p,\ \text{repeated } q-1 \text{ times}\},\end{matrix}\right]\end{aligned}$$

where $K(n,\ p,\ q)$ is defined by (3.3). It is also easy to see that the parameters satisfy the conditions for the existence of the contour C_1 and hence the G-function in (3.4) exists. In the following section, $f(v)$ will be represented in terms of simple computable functions with the help of the residue theorem.

4. Density in Series Form

Let

$$\Delta(\alpha)=\frac{\Gamma^{q-1}[\alpha+p]\prod_{j=1}^{p-1}\Gamma^{q}[\alpha+j]}{\prod_{j=1}^{pq-1}\Gamma[\alpha+p+j/pq]} \tag{4.1}$$

Then the density $f(v)$ is written as

$$f(v)=K(n,\ p,\ q)v^{n-p-1}\,(2\pi\omega)^{-1}\int_{C_1}\Delta(\alpha)^{-\alpha}d\alpha. \tag{4.2}$$

It is easy to see that the integrand has a pole at $\alpha=-i,\ i=1,\ 2,\ \cdots,$ of order a_i, which is given by

$$a_i=\begin{cases} qi, & i=1,\ 2,\ \cdots,\ p-1,\\ pq-1, & i=p,\ p+1,\cdots \end{cases} \tag{4.3}$$

Now, using the residue theorem for the right hand side of (4.2), the density as a sum of residues is given by

$$f(v)=K(n,\ p,\ q)\,v^{n-p-1}\Big(\sum_{i=1}^{\infty}R_i\Big) \tag{4.4}$$

where R_i is the residue at the pole $\alpha=-i$ of order a_i. Also, from the calculus of residue,

$$\begin{aligned} R_i&=\frac{1}{(a_i-1)!}\lim_{\alpha\to -i}\frac{\partial^{a_i-1}}{\partial\alpha^{a_i-1}}\left[(\alpha+i)^{a_i}\,\Delta(\alpha)\,v^{-\alpha}\right]\\ &=\frac{1}{(a_i-1)!}\lim_{\alpha\to -1}\frac{\partial^{a_i-1}}{\partial\alpha^{a_i-1}}\left[A_i\,v^{-\alpha}\right] \qquad (4.5)\\ &=\frac{v^i}{(a_i-1)!}\sum_{r=0}^{a_i-1}\binom{a_i-1}{r}(-\log v)^{a_i-1-r}\,A_{i0}^{(r)} \end{aligned}$$

where

$$\begin{aligned} A_i&=\frac{\Gamma^{qi}[\alpha+i+1]\Gamma^{q-1}[\alpha+p]\prod_{j=i+1}^{p-1}\Gamma^{q}[\alpha+j]}{\prod_{j=1}^{pq-1}\Gamma\left[\alpha+p+\frac{j}{pq}\right]\prod_{j=1}^{i-1}(\alpha+j)^{qj}} \quad \text{for } i=1,\ 2,\ \cdots,\ p-1\\ &=\frac{\Gamma^{pq-1}[\alpha+i+1]}{\prod_{j=1}^{pq-1}\Gamma\left[\alpha+p+\frac{j}{pq}\right]\prod_{j=1}^{p-1}(\alpha+j)^{qj}\prod_{j=p}^{i-1}(\alpha+j)^{pq-1}} \quad \text{for } i=p,\ p+1,\cdots \end{aligned}$$

$$\begin{aligned} A_{i0}&=A_i\ (\text{at } \alpha=-i)\\ &=\frac{\Gamma^{q-1}[p-i]\prod_{j=i+1}^{p-1}\Gamma^{q}[j-i]}{\prod_{j=1}^{pq-1}\Gamma\left[-i+p+\frac{j}{pq}\right]\prod_{j=1}^{i-1}(j-i)^{qj}} \quad \text{for } i=1,\ 2,\ \cdots,\ p-1 \qquad (4.6) \end{aligned}$$

(4.7) $$=\frac{1}{\prod_{j=1}^{pq-1}\Gamma\left[-i+p+\frac{j}{pq}\right]\prod_{j=1}^{p-1}(j-i)^{qi}\prod_{j=p}^{i-1}(j-i)^{pq-1}} \quad \text{for } i=p,\ p+1,\cdots$$

$$A_i^{(1)}=\frac{\partial}{\partial\alpha}A_i=A_i\frac{\partial}{\partial\alpha}\log A_i=A_i\,B_i(\text{say})$$

$$A_{i0}^{(r)}=\frac{\partial^{r-1}}{\partial\alpha^{r-1}}(A_iB_i)\ \ (\text{at }\alpha=-i)$$

$$=\sum_{m=0}^{r-1}\binom{r-1}{m}A_{i0}^{(r-1-m)}\,B_{i0}^{(m)}$$

$$B_i=\frac{\partial}{\partial\alpha}\log A_i$$

$$=qi\,\psi(\alpha+i+1)+(q-1)\,\psi(\alpha+p)+q\sum_{j=i+1}^{p-1}\psi(\alpha+j)$$

$$-\sum_{j=1}^{pq-1}\psi\left(\alpha+p+\frac{j}{pq}\right)-q\sum_{j=1}^{i-1}j(\alpha+j)^{-1} \quad \text{for } i=1,\ 2,\ \cdots,\ p-1$$

$$=(pq-1)\,\psi(\alpha+i+1)-\sum_{j=1}^{pq-1}\psi\left(\alpha+p+\frac{j}{pq}\right)-q\sum_{j=1}^{p-1}j(\alpha+j)^{-1}$$

$$-(pq-1)\sum_{j=p}^{i-1}(\alpha+j)^{-1} \quad \text{for } i=p,\ p+1,\cdots$$

$$B_{i0}=B_i\ \ (\text{at }\alpha=-i)$$

(4.8) $$=qi\,\psi(1)+(q-1)\,\psi(p-i)+q\sum_{j=i+1}^{p-1}\psi(j-i)-\sum_{j=1}^{pq-1}\psi\left(p-i+\frac{j}{pq}\right)$$

$$-q\sum_{j=1}^{i-1}j(j-i)^{-1}, \quad \text{for } i=1,\ 2,\ \cdots,\ p-1$$

(4.9) $$=(pq-1)\,\psi(1)-\sum_{j=1}^{pq-1}\psi\left(p-i+\frac{j}{pq}\right)-q\sum_{j=1}^{p-1}j(j-i)^{-1}$$

$$-(pq-1)\sum_{j=p}^{i-1}(j-i)^{-1} \quad \text{for } i=p,\ p+1,\cdots$$

$$B_i^{(m)}=\frac{\partial^{m+1}}{\partial\alpha^{m+1}}\log A_i$$

$$=(-1)^{m+1}m!\,[qi\,\zeta(m+1,\ \alpha+i+1)+(q-1)\,\zeta(m+1,\ \alpha+p)$$

$$+q\sum_{j=i+1}^{p-1}\zeta(m+1,\ \alpha+j)-\sum_{j=1}^{pq-1}\zeta\left(m+1,\ \alpha+p+\frac{j}{pq}\right)$$

$$+q\sum_{j=1}^{i-1}j(\alpha+j)^{-1-m}] \quad \text{for } i=1,\ 2,\ \cdots,\ p-1$$

$$=(-1)^{m+1}m!\,[(pq-1)\,\zeta(m+1,\ \alpha+i+1)-\sum_{j=1}^{pq-1}\zeta\left(m+1,\ \alpha+p+\frac{j}{pq}\right)$$

$$+q\sum_{j=1}^{p-1} j(\alpha+j)^{-1-m}+(pq-1)\sum_{j=p}^{i-1}(\alpha+j)^{-1-m}] \text{ for } i=p,\ p+1,\cdots$$

$$B_{i0}^{(m)}=B_i^{(m)} \text{ (at } \alpha=-i)$$

(4.10) $$=(-1)^{m+1}m![qi\,\zeta(m+1,\ 1)+(q-1)\,\zeta(m+1,\ p-i)$$

$$+q\sum_{j=i+1}^{p-1}\zeta(m+1,\ j-i)-\sum_{j=1}^{pq-1}\zeta\left(m+1,\ -i+p+\frac{j}{pq}\right)$$

$$+q\sum_{j=1}^{i-1} j(j-i)^{-1-m}] \text{ for } i=1,\ 2,\ \cdots,\ p-1$$

(4.11) $$=(-1)^{m+1}m!\ [(pq-1)\,\zeta(m+1,\ 1)-\sum_{j=1}^{pq-1}\zeta\left(m+1,\ -i+p+\frac{j}{pq}\right)$$

$$+q\sum_{j=1}^{p-1} j(j-i)^{-1-m}+(pq-1)\sum_{j=p}^{i-1}(j-i)^{-1-m}] \text{ for } i=p,\ p+1,\cdots$$

Note that $\psi(\cdot)$ is the Psi function (see Abramowitz and Stegun, 1965), and $\zeta(\cdot,\cdot)$ is the generalized Rieman Zeta function defined by $\zeta(z,\ a)=\sum_{n=0}^{\infty}(a+n)^{-z}$, $z\neq 0,\ -1,\ -2,\cdots$; where a is constant. Substituting (4.5) in (4.4), we obtain the following result:

THEOREM 4.1. *The p.d.f. of $V=\Lambda^{*1/n}$, for $n_i=n$, $i=1,\ \cdots,\ q$, where Λ^* is the modified likelihood ratio criterion for testing H defined in (1.1), is given by*

$$f(v)=K(n,\ p,\ q)\ v^{n-p-1}\left[\sum_{i=1}^{p-1}\frac{v^i}{(qi-1)!}\sum_{r=0}^{qi-1}\binom{qi-1}{r}(-\log v)^{qi-1-r}A_{i0}^{(r)}\right.$$

$$\left.+\sum_{i=p}^{\infty}\frac{v^i}{(pq-2)!}\sum_{r=0}^{pq-2}\binom{pq-2}{r}(-\log v)^{pq-2-r}\,A_{i0}^{(r)}\right]$$

where $K(n,\ p,\ q)$ and $A_{i0}^{(r)}$ are given by (3.3) and (4.6)-(4.11) respectively.

By integrating above expression term by term, from 0 to v (<1), we get the corresponding cummulating distribution. When $q=1$, we get the distribution of the sphericity criterion as a corollary of the above theorem.

COROLLARY 4.1. *The p.d.f. of $V=\Lambda^{*1/n}$, when $n=N-1$ and $\Lambda^*$is the modified likelihood ratio criterion for testing $H:\ \Sigma=\sigma^2 I_p$ in the complex normal population, is given by*

$$f(v)=K(n,\ p)\ v^{n-p-1}\left[\sum_{i=1}^{p-1}\frac{v^i}{(i-1)!}\sum_{r=0}^{i-1}\binom{i-1}{r}\ (-\log v)^{i-1-r}\,A_{i0}^{(r)}\right.$$

$$\left.+\sum_{i=p}^{\infty}\frac{v^i}{(p-2)!}\sum_{r=0}^{p-2}\binom{p-2}{r}(-\log v)^{p-2-r}\,A_{i0}^{(r)}\right],$$

where $K(n, p)$ *and* $A_{i0}^{(r)}$ *are obtained from (3.3) and (4.6)-(4.11) by substituting* $q=1$.

5. Asymptotic Expansion of the Distribution

In this section we develop a large sample distribution theory for a suitable function of Λ^* as developed by Box and given in Anderson (1958). Let $Y=-2\rho \log \Lambda^*$, where ρ is an arbitrary constant to be determined later. Using (2.2) the characteristic function of Y is given by

$$(5.1)\qquad \phi_Y(t)=K\frac{(n_0 p)^{-2\omega t\rho p n_0}}{\prod_{i=1}^{q}\prod_{j=1}^{p}(k_i n_0)^{-2\omega t\rho k_i n_0}}\frac{\prod_{j=1}^{p}\prod_{i=1}^{q}\Gamma[k_i n_0(1-2\omega t\rho)+1-j]}{\Gamma[p n_0(1-2\omega t\rho)]}$$

where K is a normalizing constant, and $k_i=n_i/n_0$. Now expand $\phi_Y(t)$ using Barnes' expansion formula for the gamma function in terms of Bernoulli polynomials, (see Anderson, 1958, p. 204)

$$(5.2)\qquad \log \phi_Y(t)=-\frac{f}{2}\log(1-2\omega t)-\sum_{r=1}^{m}\omega_r[(1-2\omega t)^{-r}-1]+0(n_0^{-1-m}),$$

where $f=p^2 q-1$,

$$(5.3)\qquad \omega_r=\frac{(-1)^{r+1}}{r(r+1)(\rho n_0)^r}\left[\sum_{j=1}^{p}\sum_{i=1}^{q}\frac{B_{r+1}[(1-\rho)k_i n_0+1-j]}{k_i^r}-\frac{B_{r+1}[(1-\rho)p n_0]}{p^r}\right]$$

and $B_r(\cdot)$ is the Bernoulli polynomial of degree r and order unity. Now the constant ρ is determined such that $\omega_1=0$, and we obtain

$$(5.4)\qquad \rho=1-\frac{1}{6n_0 f}\left[\left(\sum_{i=1}^{q}\frac{1}{k_i}\right)p(2p^2-1)-\frac{1}{p}\right].$$

Denote the value of ω_2 for this value of ρ by γ_2 and calculate it from (5.4) using the result $B_3(a)=a^3-\frac{3}{2}a^2+\frac{1}{2}a$. Substituting these values in (5.2) and inverting the characteristic function we obtain the following result.

THEOREM 5.1. *The c.d.f. of* $-2\rho \log \Lambda^*$ *where* Λ^* *is the modified likelihood ratio criterion for testing* H, *is given by*

$$P(-2\rho \log \Lambda^* \leq u)=P(\chi_f^2 \leq u)-\gamma_2[P(\chi_{f+4}^2 \leq u)-P(\chi_f^2 \leq u)]+0(n_0^{-3}),$$

where $f=p^2q-1$, ρ *given by* (5.4), *and* χ_ν^2 *denotes a chi-square random variable with* ν *d.f.*

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