Jordan Automorphisms on Direct Sums of Simple Rings

R.A. Heeg

1. Introduction

Suppose \( R \) is a direct sum of \( K \) simple rings and \( G \) is a group of automorphisms of \( R \) of finite order, \(|G|\). If \( R \) has no \(|G|\)-torsion (i.e. \(|G|r=0 \) implies \( r=0 \) for all \( r \in R \)) then Osterburg has shown in [6] that the fixed ring of \( R \) under \( G \) is a direct sum of at most \(|G|K\) simple rings. We shall prove the analogous result when \( G \) consists of Jordan automorphisms of \( R \).

2. Preliminaries

Let \( R \) and \( S \) be rings and \( T \) an additive map of \( R \) into \( S \). Then \( T \) is called a Jordan homomorphism if (i) \((x^2)T=(xT)^2\) and (ii) \((xyx)^T=xTyTzT\) for all \( x, y \) in \( R \). Any additive map satisfying (i) necessarily satisfies (i') \((xy+yx)^T=xTyT+yTxT\) and if \( S \) has no 2-torsion (i.e. \( 2s=0 \) implies \( s=0 \) for every \( s \in S \)) then additivity and (i') imply both (i) and (ii) [c.f. Herstein: Topics in Ring Theory].

As can be readily verified, any Jordan homomorphism \( T \) also satisfies \((xyz+zxy)^T=xTyTzT+zTyTzT\) as well as \([x, [y, z]]T=[xT, [yT, zT]]\) where \([a, b]=ab-ba\).

Clearly every (associative) homomorphism or anti-homomorphism is a Jordan homomorphism and conversely we have

**Theorem 1.** (Herstein [1]) Every Jordan homomorphism onto a prime ring is either a homomorphism or anti-homomorphism.

As a corollary we have

**Corollary 2.** (Martindale-Montgomery [4]) Let \( T \) be a Jordan isomorphism from \( R \) onto \( S \) and let \( P \) be a prime ideal of \( R \). Then the image of \( P \) under \( T \) (denoted \( PT \)) is a prime ideal of \( S \) and \( R/P \),

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Suppose $R$ is a non-commutative ring with involution $\ast$. Define $\tau : R \to R \oplus R$ by $r' = (r, r^\ast)$. Then $\tau$ is a Jordan monomorphism (i.e. $\tau$ is a one-to-one Jordan homomorphism) whose image is not a subring of $R \oplus R$. Similarly, there are Jordan homomorphisms whose kernels are not (associative) ideals. This prompts the following definitions.

An additive subgroup $A$ of an associative ring $R$ is called a (special) Jordan ring if $a^2, aba$ are in $A$ whenever $a, b \in A$.

An additive subgroup $I$ of a Jordan ring $A$ is called a (quadratic) Jordan ideal if $x^2, xax, axa, xa + ax$ are in $I$ whenever $x \in I$ and $a \in A$. We write $I \subseteq_j A$.

Every associative ring is a Jordan ring and every ideal of an associative ring is a Jordan ideal. The image of a Jordan homomorphism is a Jordan ring and the kernel of a Jordan homomorphism is a Jordan ideal. Also, the image of a Jordan ideal under a Jordan homomorphism is a Jordan ideal of the image of the Jordan homomorphism.

In [3], McCrimmon has shown that every non-zero Jordan ideal of a semiprime ring contains a nonzero (associative) ideal. Using this fact and Corollary 2, we can characterize Jordan automorphisms on direct products of prime rings. But first we give some examples.

**Example 1.** Let $R$ be a commutative ring and $\text{Mat}_n(R)$ denote the $n \times n$ matrices over $R$. Then the map $M \to M^\ast$ which takes each matrix to its transpose is an involution and hence a Jordan automorphism.

**Example 2.** Let $R$ be a non-commutative simple ring with involution $\ast$. Define $\tau : R \oplus R \to R \oplus R$ by $\tau(a, b) = (a^\ast, b)$. Then $\tau$ is a Jordan automorphism (of order 2) on a direct sum of simple rings which is neither an automorphism nor anti-automorphism.

**Example 3.** Let $R, \ast$ be as in example 2. Define $\tau : R \oplus R \to R \oplus R$ by $\tau(a, b) = (b, a^\ast)$. Then $\tau$, is a Jordan automorphism of order 4.

**Example 4.** Let $R, \ast$ be as in example 2. Let $\rho$ be a prime greater than 2. Let $S = R_1 \oplus R_2 \oplus \ldots \oplus R_p$ each $R_i = R$. Define $\tau_\rho : S \to S$ by $\tau_\rho(r_1, r_2, \ldots, r_p) = (r_2^\ast, r_3^\ast, \ldots, r_p^\ast, r_1)$; then $\tau_\rho$ is a Jordan automorphism of order $\rho$.

We note that in examples 2, 3, 4 we can replace $R$ with any ring and $\ast$ with any Jordan automorphism on $R$ in order to obtain a Jordan
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The following theorem shows that these are the only Jordan automorphisms on direct products of prime rings.

**Theorem 3.** Let \( R = \prod_{\alpha \in \Lambda} S_\alpha \) where each \( S_\alpha \) is a prime ring and let \( g \) be a Jordan automorphism on \( R \). Then for each \( \beta \in \Lambda \), there is \( \gamma \in \Lambda \) so that \( S_\beta^g = S_\gamma \) and the restriction of \( g \) to \( S_\beta \) is either an automorphism or anti-automorphism.

**Proof.** \( S_\beta \) is the intersection of the prime ideals of \( R \) which contain \( S_\beta \). Therefore the image of \( S_\beta \) under \( g \) is an intersection of prime ideals of \( R \) by Corollary 2. In particular, \( S_\beta^g \) is an associative ideal of \( R \).

Let \( K \subseteq \Lambda \) such that \( S_\beta^g \) contains an element which has a non-zero entry in \( S_\beta \) if and only if \( \ell \in K \).

Suppose \( \ell \in K \) and \( s \in S_\ell \) such that \( s \) appears in the \( l^{th} \) component of an element of \( S_\beta^g \) with \( s \neq 0 \). Since \( S_\beta^g \) is an ideal of \( R \), \( S_\beta^g S_\ell \subseteq S_\beta^g \). Consequently \( s S_\ell \subseteq S_\beta^g \). Likewise \( S_\ell s \subseteq S_\beta^g \) and \( x s y \in S_\beta^g \) for all \( x, y \in S_\ell \). Therefore \( S_\beta^g \) contains an ideal of \( S_\ell \), which is non-zero by the primeness of \( S_\ell \).

So for each \( \ell \in K \) there is a nonzero ideal \( I_\ell \) of \( S_\ell \) such that \( S_\beta^g \supseteq \prod_{\ell \in K} I_\ell \).

We now show that \( K \) contains exactly one element. Suppose \( \ell, \ell' \in K \), \( \ell \neq \ell' \). Then \( I_\ell \cap I_{\ell'} = 0 \). Thus \( 0 = g^{-1}(I_\ell \cap I_{\ell'}) = g^{-1}(I_\ell) \cap g^{-1}(I_{\ell'}) \).

But \( g^{-1}(I_\ell), g^{-1}(I_{\ell'}) \) are nonzero Jordan ideals of \( S_\beta \). By McRimmon's result there are nonzero ideals \( A \) and \( B \) of \( S_\beta \) with \( A \subseteq g^{-1}(I_\ell) \) and \( B \subseteq g^{-1}(I_{\ell'}) \). But this forces \( A \cap B = 0 \) which contradicts the primeness of \( S_\beta \). Thus \( K \) contains exactly one element. This implies that there is a \( \gamma \in \Lambda \) such that \( S_\beta^g \subseteq S_\gamma \).

Applying the same argument to \( S_\gamma \) and \( g^{-1} \), we get \( S_\gamma^{g^{-1}} \subseteq S_\beta \) where \( \delta \in \Lambda \). But this implies that \( S_\gamma \subseteq S_\beta^g \) and so \( S_\beta^g \subseteq S_\gamma \subseteq S_\delta^g \) which gives \( S_\beta \subseteq S_\delta \) forcing \( \beta = \delta \). Consequently \( S_\beta^g = S_\gamma \).

The last statement of the theorem is a consequence of Theorem 1.

If \( R \) is a ring and \( G \) is a group of Jordan automorphisms of \( R \), then the fixed ring of \( R \) under \( G \) is \( \{ r \in R \mid r^g = r \text{ for every } g \in G \} \) and is denoted \( R^G \). If \( g \) is an element of any group, then \( \langle g \rangle \) denotes the subgroup generated by \( g \). In particular, if \( I \) is a subring of \( R \) which is \( g \)-invariant (i.e. \( I^g = I \)) then \( I^{<g>} \) denotes the set of elements of \( I \) which are fixed by \( g \).
Finally, we note another consequence of Theorem 1 which appears in [4]. Namely, if \( R \) is a prime ring, \( G \) is a group of Jordan automorphisms of \( R \), and \( H \) is the subgroup of \( G \) consisting of (associative) automorphisms of \( R \), then the index of \( H \) in \( G \) is either one or two. In either case, \( H \) is normal in \( G \). If \( H \neq G \) then \( G/H \) acts as involution on \( R^H \).

### 3. Main Theorem

In this section we consider the action of a finite group of Jordan automorphisms on a ring which is a direct sum of simple rings. Our main result (Theorem 11) extends theorems of both Osterburg [6] and Sundstrom [7]. We start with the result in [6].

**Theorem 4 [Osterburg].** Let \( R \) be a ring which is the direct sum of \( k \) simple rings and \( G \) a finite group of automorphisms of \( R \) such that \( R \) has no \( |G| \)-torsion. Then the fixed ring of \( R \) is a direct sum of \( l \) simple rings where \( l \leq k|G| \).

For involutions, we have the following result proved in [1].

**Theorem 5.** If \( R \) is a simple ring of characteristic not 2 and \( g \) is an involution on \( R \), then the fixed ring of \( R \) is a simple Jordan ring.

A simple Jordan ring is a Jordan ring which has no nonzero proper Jordan ideals. Since every associative ideal is a Jordan ideal, any associative ring which is a simple Jordan ring is a simple ring. Conversely, if \( R \) is a simple ring then \( R \) is a simple Jordan ring. For if \( R \) has a nontrivial proper Jordan ideal, \( A \), then by McCrimmon's result \( R \) contains a nonzero associative ideal contained in \( A \), which contradicts the simplicity of \( R \).

In [7], Sundstrom considers the situation when \( G \) is a finite solvable group consisting of automorphisms or anti-automorphisms, acting on a direct sum of simple rings which has no \( |G| \)-torsion. The subgroup of automorphisms of \( G \) is a normal subgroup, \( H \), of index 2 with \( G/H \) acting on \( R^H \) as an involution. In general, when \( G \) is a finite solvable group of Jordan automorphisms on a direct sum of simple rings, the subgroup of automorphisms is not necessarily of index 2 in \( G \). In example 3, \( \tau \) is a Jordan automorphism of order 4 and the only automorphism in \( \langle \tau \rangle \) is the identity. When \( G \) is not solvable, the subgroup of automorphisms is not necessarily normal as the next
example illustrates.

**Example 5.** Let \( R \) be a simple non-commutative ring with involution \( * \). Let \( S = R \circledast R \circledast R \) and \( G = \langle \tau, \rho \rangle \) where \( \tau(a, b, c) = (a, c^*, b) \) and \( \rho(a, b, c) = (c, a, b) \) then \( \tau \rho^{-1}(a, b, c) = (b^*, c^*, a) \) so \( \tau \rho \rho^{-1} \) is not an automorphism and hence the subgroup of automorphisms of \( G \) is not normal.

Nevertheless, by using Theorems 3, 4, and 5, we can prove analogous results for Jordan automorphisms.

We start with

**Lemma 6.** Suppose \( R = \sum_{i=0}^{n-1} \oplus S_i \) and \( g \) is a Jordan automorphism of \( R \) such that

(i) \( g^n \) is the identity

and

(ii) \( S_i^g = S_{i+1(mod \ n)} \)

Then the fixed ring of \( R \) is Jordan isomorphic to \( S_0 \).

**Proof.** If \( s \in S_0 \) then \( s \oplus s^g \oplus s^{2g} \oplus \cdots \oplus s^{ng-1} \) is fixed by \( g \). Conversely, if \( r \in R \) is fixed by \( g \) then \( r \) is of the form \( s \oplus s^g \oplus s^{2g} \oplus \cdots \oplus s^{ng-1} \) where \( s \in S_0 \). So \( R^{<g^*>} = \{ s \oplus s^g \oplus s^{2g} \oplus \cdots \oplus s^{ng-1} | s \in S_0 \} \). The map from \( S_0 \) to \( R^{<g^*>} \) given by \( s \rightarrow s \oplus s^g \oplus s^{2g} \oplus \cdots \oplus s^{ng-1} \) is a Jordan isomorphism.

We generalize the result in

**Lemma 7.** Suppose \( R \) is a ring and \( g \) is a Jordan automorphism such that \( R = \sum_{i=0}^{n-1} \oplus I^i \). Then the fixed ring of \( R \) is Jordan isomorphic to the fixed ring of \( I \) under \( \langle g^n \rangle \).

**Proof.** Clearly, each \( I^i \) is \( g^n \)-invariant so

\[
R^{<g^*>} = \bigoplus_{i=0}^{n-1} I^i <g^*> = \bigoplus_{i=0}^{n-1} (I^i)^{<g^*>}
\]

By letting \( S_i = (I^i)^{<g^*>} \) and \( g' \) be a generator of \( \langle g \rangle / \langle g^n \rangle \) we can apply Lemma 6 to \( \bigoplus_{i=0}^{n-1} S_i \) and \( g' \) to obtain

\[
(\bigoplus_{i=0}^{n-1} (I^i)^{<g^*>})^{<g'/g^n>} = (\bigoplus_{i=0}^{n-1} S_i)^{<g'>} = S_0 = I^{<g^n>}.
\]

But

\[
(\bigoplus_{i=0}^{n-1} (I^i)^{<g^*>})^{<g>/g^n>} = (R^{<g^*>})^{<g>/g^n} = R^{<g^*>}
\]

So \( R^{<g^*>} \cong g I^{<g^*>} \).
We now prove

**Theorem 8.** Let $R$ be a simple ring and $G$ a finite group of Jordan automorphisms of $R$. If $R$ has no $|G|$-torsion, then the fixed ring of $R$ is a direct sum of at most $|G|$ simple Jordan rings. If, in addition, $G$ does not consist solely of automorphisms then the fixed ring of $R$ is a direct sum of at most $|G|/2$ simple Jordan rings.

**Proof.** Let $H = \{ g \in G | g \text{ is an automorphism of } R \}$. If $H = G$, then by Theorem 4 we are done.

We now assume $H \neq G$. Then the index of $H$ in $G$ is equal to 2 and $G/H$ acts as involution of $R^H$. By Theorem 4, $R^H$ is a direct sum of at most $|H|$ simple rings; so suppose $R^H = \bigoplus_{i=1}^{n} S_i$ where each $S_i$ is a simple ring and $n \leq |H|$.

We first consider the case when $S_1$ is $G/H$-invariant. Either the action of $G/H$ on $S_1$ is that of the identity or that of an involution. In either case, $S_1^{G/H}$ is a simple Jordan ring.

Now suppose $S_1$ is not $G/H$ invariant. Then by Theorem 3, there is an $1 \leq n$ that the image of $S_1$ under the non-identity element of $G/H$ is $S_1$. In this case $G/H$ acts on $S_1 \oplus S_1$ and by Lemma 6, $(S_1 \oplus S_1)^{G/H} \cong S_1$. Continuing, we see that $R^G = (R^H)^{G/H}$ is a direct sum of at most $|H| = |G|/2$ simple Jordan rings.

We now investigate the situation when $R$ is a direct sum of simple rings, proving first a result about associative automorphisms.

**Lemma 9.** Let $R = S_1 \oplus S_2$ where $S_1, S_2$ are simple rings and let $G$ be a finite group of automorphisms of $R$. If $R$ has no $|G|/2$ torsion and $S_1$ is not $G$-invariant, then the fixed ring of $R$ is a direct sum of at most $|G|/2$ simple rings.

**Proof.** Let $K = \{ g \in G | S_1^g = S_1 \}$. Then $K$ is normal in $G$ and has index 2. Consequently,

$$R^G = (R^K)^{G/K} = ((S_1 \oplus S_2)^K)^{G/K} = (S_1^K \oplus S_2^K)^{G/K}$$

with is isomorphic to $S_1^K$ by Lemma 6. And by Theorem 4, $S_1^K$ is a direct sum of at most $|K|$ simple rings.

Therefore, $R^G$ is a direct sum of at most $|G|/2$ simple rings.

We now extend this Lemma by allowing Jordan automorphisms.
THEOREM 10. Let \( R = S_1 \oplus S_2 \) where \( S_1, S_2 \) are simple rings and let \( G \) be a finite group of Jordan automorphisms of \( R \) such that \( R \) has no \(|G|\)-torsion.

(i) If \( G \) does not consist solely of automorphisms, then the fixed ring is a direct sum of at most \( 3|G|/2 \) simple Jordan rings.

(ii) If \( S_1 \) is not \( G \)-invariant, then the fixed ring is a direct sum of at most \( |G|/2 \) simple Jordan rings.

(iii) If the hypotheses of (i) and (ii) are both satisfied and \( \{g \in G | S_1^g = S_1\} \neq \{g \in G | g \text{ is an automorphism}\} \), then the fixed ring is a direct sum of at most \( |G|/4 \) simple Jordan rings.

Proof. Let \( K = \{g \in G | S_1^g = S_1\} \) and let \( H = \{g \in G | g \text{ is an automorphism of } R\} \). We first prove:

(ii) Suppose \( S_1 \) is not \( G \)-invariant. Then the index of \( K \) in \( G \) is equal to 2. As in the proof of Lemma 9, \( R^G \cong \oplus S_1^K \) and by Theorem 8, \( S_1^K \) is a direct sum of at most \( |K| = |G|/2 \) simple Jordan rings.

(i) We may assume that \( S_1 \) is \( G \)-invariant. Otherwise, we can apply part (ii). Since \( G \) does not consist solely of automorphisms, its action on either \( S_1 \) or \( S_2 \) is not that of associative automorphisms. Therefore the fixed ring of either \( S_1 \) or \( S_2 \) is a direct sum of at most \( |G|/2 \) simple Jordan rings by Theorem 8. The fixed ring of the other summand is a direct sum of at most \( |G| \) simple Jordan rings also by Theorem 8. Therefore \( R^G \) is a direct sum of at most \( |G|/2 + |G| = 3|G|/2 \) simple Jordan rings.

(iii) As in the proof of Lemma 9, \( R^G \cong \oplus S_1^K \) (or, equivalently, \( R^G \cong \oplus S_2^K \)). If the action of \( K \) on both \( S_1 \) and \( S_2 \) is that of automorphisms then \( K \subseteq H \). But the index of \( K \) in \( G \) is equal to 2. So either \( K = H \) or \( H = G \). But, by hypothesis, neither of these can happen. Consequently, we may assume that \( K \) does not act as automorphism on \( S_1 \). By applying Theorem 8, \( S_1^K \) is a direct sum of at most \( |K|/2 \) simple Jordan rings. That is, \( R^G \) is a direct sum of at most \( |K|/2 = |G|/4 \) simple Jordan rings.

We remark that when \( S_1 \) is not \( G \)-invariant, we need only require that \( R \) has no \(|G|/2\) torsion.

As a final result we prove:

THEOREM 11. Let \( R \) be a direct sum of \( K \) simple rings and \( G \) a finite group of Jordan automorphism of \( R \) such that \( R \) has no \(|G|\)-torsion. Then the fixed ring is a direct sum of at most \( K|G| \) simple
Jordan rings. This bound can be achieved only when each summand of \( R \) is \( G \)-invariant and \( G \) consists solely of automorphisms of \( R \).

Proof. Let \( R = \sum_{i=1}^{K} S_i \). If each \( S_i \) is \( G \)-invariant then we can apply Theorem 8 to conclude that \( R^G \) is a direct sum of at most \( K|G| \) simple Jordan rings. If, in addition, \( G \) does not consist solely of automorphisms, then the action of \( G \) on some \( S_i \) does not act as automorphisms. Consequently, \( S_i^G \) is a direct sum of at most \( |G|/2 \) simple rings. Therefore \( R^G \) is a direct sum of at most \((K-1)|G| + |G|/2 < K|G|\) simple Jordan rings.

Now assume that some \( S_i \) is not \( G \)-invariant and let \( \text{Orbit}(S_i) = \{S_i^s | g \in G \} \). We will show that if \( R' \) is the direct sum of the distinct elements of \( \text{Orbit}(S_i) \) then the fixed ring \( R' \) under \( G \) is a direct sum of at most \( |G|/n \) simple Jordan rings where \( n = |\text{Orbit}(S_i)| \).

Let \( H = \{g \in G | S_i^g = S_i \} \) and let \( g_0, g_1, ..., g_{n-1} \) be distinct representatives of the right cosets of \( h \) in \( G \) (where \( g_0 \) is the identity) then \( R' = S_i^{g_0} \oplus S_i^{g_1} \oplus ... \oplus S_i^{g_{n-1}} \) and \( n = [G : H] \).

We claim that \((R')^G = \{s + s^g + ... + s^{g_{n-1}} | s \in S_i^H \}\). Clearly any element of \((R')^G\) is of the form \( s + s^g + ... + s^{g_{n-1}} \) where \( s \in S^H \). Now let \( g \in G \). Then \( (s + s^g + ... + s^{g_{n-1}})^g = s^g + s^{g_1g} + ... + s^{g_{n-1}g} \). But there is a \( h \in H \) and \( i_0 (0 \leq i_0 \leq n-1) \) so that \( g = h g_{i_0} \). Consequently, \( s^g = s^{g_0 g_{i_0}} = s^{g_{i_0}} \). Similarly, there is \( h' \in H \) and \( i_1 (0 \leq i_1 \leq n-1) \) so that \( g_{i_1} = h' g_{i_1} \). Therefore \( s^{g_{i_0}} = s^{g_0 g_{i_0}} = s^{g_{i_0}} \). Continuing, we see that the action of \( g \in G \) simply permutes the elements of \( s + s^g + ... + s^{g_{n-1}} \). We need only show that \( \{g_{i_0}, g_{i_1}, ..., g_{i_{n-1}} \} \) are distinct representatives of the right cosets of \( H \) in \( G \).

Suppose \( g_{i_a} \) and \( g_{i_b} \) are in the same right coset. Then there is a \( g_j (0 \leq j \leq n-1) \) and \( h_1, h_2 \in H \) so that \( g_{i_a} = h_1 g_j \) and \( g_{i_b} = h_2 g_j \). From before, there exists \( h', h'' \) so that \( g_{a g} = h' g_{i_a} \) and \( g_{b g} = h'' g_i \) where \( g_a, g_b \) are distinct in \( \{g_0, g_1, ..., g_{n-1} \} \). Consequently, \( g_{a g} = h' g_{i_a} = h' h_1 g_j \) and \( g_{b g} = h'' g_{i_b} \). That is, \( g_a = h' h_1 g_j g^{-1} \) and \( g_b = h'' h_2 g_j g^{-1} \) which puts \( g_a \) and \( g_b \) in the same right coset of \( H \) in \( G \), a contradiction.

Thus, \( \{g_{i_0}, g_{i_1}, ..., g_{i_{n-1}} \} \) are distinct representatives of the right cosets of \( H \) in \( G \).

We have shown that \((R')^G = \{s + s^g + ... + s^{g_{n-1}} | s \in S^H \}\).

But the mapping of \( S_i^H \) onto \((R')^G \) given by \( s \rightarrow s + s^g + ... + s^{g_{n-1}} \) is a
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Jordan isomorphism. That is, \((R')^G \cong_j S^H_i\). But by Theorem 8, \(S^H_i\) is a direct sum of at most \(|H| = |G|/n\) simple Jordan rings. This completes the proof.

References


Northern Illinois University
Dekalb, Illinois 60115
U.S.A.