ON CHARACTERS OF $\eta$-RELATED TENSORS IN COSYMPLECTIC AND SASAKIAN MANIFOLDS (2)*

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0. Introduction

A $(2n+1)$-dimensional differentiable manifold $M$ is called to have a cosymplectic structure if there is given a positive definite Riemannian metric $g_{ij}$ and a triplet $(\varphi_k^i, \xi^j, \eta_k)$ of $(1,1)$ type tensor field $\varphi_k^i$, vector field $\xi^j$ and 1-form $\eta_k$ in $M$ which satisfy the following equations

\begin{align}
\varphi_j^k \varphi_i^h &= -\gamma_{jh}^{\ h}, \quad \varphi_j^i \xi^j = 0, \quad \eta_i \varphi_j^i = 0, \quad \eta_i \xi^i = 1, \\
g_{ij} \varphi_j^k \varphi_i^t &= \gamma_{ji}, \quad \eta_i = g_{ihk} \xi^h,
\end{align}

where

\begin{align}
\gamma_{ji} &= g_{ji} - \eta_j \eta_i, \quad \gamma_{j}^{\ h} = g^{ht} \gamma_{tj},
\end{align}

and

\begin{align}
\nabla_k \varphi_j^i &= 0, \quad \nabla_k \eta_j = 0,
\end{align}

where $\nabla_k$ indicates the covariant differentiation with respect to $g_{ij}$. By virtue of the last equation of (0.1), we shall write $\gamma^h$ instead of $\xi^h$ in the sequel. The indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n+1\}$. In the present paper, we define an $\eta$-projective vector field $v^h$ in a cosymplectic manifold $M$ by the condition

\begin{align}
\mathcal{L}_{v^h} \left[ j_i \right] &= \nabla_j v^h + v^t K_{tji}^{\ h} = p_{j^i}^{\ h} + \rho_i \gamma_{ji}^{\ h}
\end{align}

for a certain covector field $\rho_i$, where $\left\{ j_i \right\}$, $K_{tji}^{\ h}$ and $\mathcal{L}_{v^i}$ are the Christoffel symbols formed with $g_{ij}$, the curvature tensor and the Lie derivation with respect to $v^i$ on $M$ respectively.

The purpose of the present paper is to investigate the properties of $\eta$-projective vector fields in a compact cosymplectic manifold.

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1. \(\eta\)-projective vector fields in a cosymplectic manifold

In a \((2n+1)\)-dimensional cosymplectic manifold \(M\) with the cosymplectic structure \((\varphi, \eta, g)\), we easily obtain the following relations.

\[
(1.1) \quad K_{kij}^i \eta_i = 0, \quad K_{ji}^i \eta_j = 0
\]

by virtue of the Ricci identity with respect to \(\eta^i\), where \(K_{ji}\) is the Ricci tensor of \(M\). Moreover using the Ricci identity with respect to \(\varphi_i^h\), we easily see that \(K_{jil}^i \varphi_i^h + K_{lij}^i \varphi_i^h = 0\) and from which,

\[
(1.2) \quad K_{jil}^i \varphi_i^h + K_{lij}^i \varphi_i^h = 0.
\]

Since

\[
(1.3) \quad K_{tjis}^i \varphi_i^h = \frac{1}{2} (K_{tjis} - K_{sjit}) \varphi_i^h = -\frac{1}{2} K_{tjis} \varphi_i^h,
\]

we obtain

\[
(1.4) \quad \varphi_i^h K_{tjis}^h = 2K_{jil}^i \varphi_i^h.
\]

In a previous paper (Eum, [1]), we proved that if \(M\) is a cosymplectic manifold of constant curvature with respect to \(\gamma_{ji}\), then the curvature tensor of \(M\) is of the form:

\[
(1.5) \quad K_{kij}^h = \frac{K}{2n(2n-1)} (\gamma_k^i \gamma_{ji} - \gamma_j^i \gamma_{ki}),
\]

\(K\) being the constant scalar curvature.

If we substitute (1.5) into (0.4), we obtain

\[
(1.6) \quad \nabla_h \nabla_j \gamma^i + \frac{K}{2n(2n-1)} \varphi_i^h (\gamma_k^i \gamma_{kj} - \gamma_k^j \gamma_{ki}) = \gamma_k^i \gamma_h^j + \gamma_j^i \gamma_k^h.
\]

In this place, we consider a system of partial differential equation

\[
(1.7) \quad \nabla_h \nabla_j \gamma^i + \frac{K}{2n(2n-1)} (2\gamma_{jh} \gamma_k^j + \gamma_{kh} \gamma_j^k + \gamma_{kj} \gamma_p^h) = 0
\]

which is obtained by the substitution into (1.6) of

\[
(1.8) \quad \varphi^h = -\frac{n(2n-1)}{K} \gamma^h.
\]

The integrability condition of (1.7) is given by

\[
(1.9) \quad \nabla_h \nabla_j (\nabla_h \nabla_j \gamma^i) - \nabla_j \nabla_h (\nabla_h \nabla_j \gamma^i) = -K_{ijk}^i \nabla_j \gamma_h^i - K_{lij}^i \nabla_k \gamma_i^h - K_{lik}^i \nabla_j \gamma_k^h.
\]

If we assume that \(\gamma_i^h \gamma_i^j = 0\), then the condition (1.9) is satisfied by
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(1.5) and (1.7). In this case we obtain

\[
\mathcal{L}_p \left[ \frac{h}{k_j} \right] = \nabla_h \nabla_j \rho^h + \rho^h K_{kji}^h = -\frac{K}{n(2n-1)} (\gamma^h p_j + \gamma^j p_k)
\]

by virtue of (1.5) and (1.7), where \( \mathcal{L}_p \) indicates the Lie derivation with respect to \( \rho^h \).

Thus we have the following

**Theorem 1.1.** Let \( M \) be a cosymplectic manifold of constant curvature with respect to \( \gamma_{ji} \). In this case, if \( \rho^i \) in \( M \) belongs to the distribution orthogonal to \( \eta \), that is, \( \rho_i \eta^j = 0 \), then \( \rho^i \) is an \( \eta \)-projective vector field locally and the associated vector of \( \rho^i \) is given by \( -\frac{K}{n(2n-1)} \rho^i \), \( K \) being the constant scalar curvature.

By contractions in (0.4), find

(1.11) \[ \nabla_j \nabla_i \mathcal{v}^i = (2n+1) \rho_j - (\rho_i \eta^j) \eta_i \]

and

(1.12) \[ \nabla_i \nabla_j \mathcal{v}^i = \mathcal{v}^i K_{ji} + (2n+1) \rho_i - (\rho_i \eta^j) \eta_i. \]

Transvecting (0.4) with \( \eta_h \) and taking account of (0.2), (0.3) and (1.1), we easily see that

(1.13) \[ \nabla_j \nabla_i (\mathcal{v}^i \eta_i) = 0. \]

Therefore in a compact orientable cosymplectic manifold \( M \), we obtain

(1.14) \[ \mathcal{v}^i \eta_i = c \]

c being a constant (Yano, [5]), and from which

(1.15) \[ \mathcal{L}_c \eta_i = 0. \]

Substituting (0.4) into the formula (Yano [5])

\[
(1) \quad \mathcal{L}_c K_{kji}^h = \nabla_k \mathcal{L}_c \left[ \frac{h}{ji} \right] - \nabla_j \mathcal{L}_c \left[ \frac{h}{ki} \right]
\]

we obtain

(1.16) \[ \mathcal{L}_c K_{kji}^h = (\nabla_k \rho_j - \nabla_j \rho_k) \gamma^h_{ji} + (\nabla_k \rho_i) \gamma^h_{ji} - (\nabla_j \rho_i) \gamma^h_{ki}, \]

and from which

(1.17) \[ \mathcal{L}_{\mathcal{v}} K_{ji} = \nabla_i \rho_j - 2n \nabla_j \rho_i - \eta^i \left( (\nabla_i \rho_j) \eta_j + (\nabla_i \rho_i) \eta_j \right) + \eta^i (\nabla_j \rho_i) \eta_i. \]

Using the relation \( \mathcal{L}_{\mathcal{v}} K_{ji} = \mathcal{L}_{\mathcal{v}} K_{ij} \), we obtain
(1.18) \((2n+1)(V_i p_j - V_j p_i) = \eta^i \{(V_i p_i) \eta_j - (V_j p_j) \eta_i \}\).

Transvecting (1.18) with \(\eta^i\), we obtain

(1.19) \((2n+1) \eta^i V_i p_j - 2n V_j (p_i \eta^i) = \mu \eta_j\),

where we have put

(1.20) \(\mu = \eta^i (V_i p_i) \eta^i\).

We consider on the case that \(M\) is a compact cosymplectic manifold. Taking account of the second equation of (1.1) and (1.15), we obtain

(1.21) \(\eta_h \mathcal{L}_v K^h = \eta_h \mathcal{L}_v(K_h \mathcal{G}^{th}) = 0\).

Substituting (1.17) into (1.21), we obtain

(1.22) \((2n-1) V_h (p \eta^h) = \eta_h K_{ht} \mathcal{G}^{th}\)

by virtue of (1.15).

Transvecting (1.22) with \(\eta^h\) and taking account of (1.1) and (1.20), we see that

(1.23) \(\mu = 0\).

Substituting (1.23) into (1.19), we obtain

(1.24) \((2n+1) \eta^i V_i p_j = 2n V_j p_i\),

where we have put

(1.25) \(\rho = p_i \eta^i\).

Substituting (1.16) into the equation

\(\varphi^{st} [(\mathcal{L}_v K^{st}) \eta^j + K_{ts}^h \mathcal{L}_v \eta^j] = 0\),

which is obtained from the first equation of (1.1), and taking account of (0.1), (1.25) and the fact that \(\varphi^{st} = -\varphi^{st}\), we obtain

(1.26) \(\varphi^{st} K_{ts}^h \mathcal{L}_v \eta^j = 2 \varphi^{st} V_i \rho\).

Substituting (1.17) into the equation which is obtained from the second equation of (1.1)

(1.27) \(2 \varphi^{st} [(\mathcal{L}_v K^{st}) \eta^j + K_{st} \mathcal{L}_v \eta^j] = 0\)

and taking account of (1.23) and (1.25), we obtain

(1.28) \(2 \varphi^{ht} K_{jt} \mathcal{L}_v \eta^j = 2(2n-1) \varphi^{ht} V_i \rho\).

Taking account of (1.3), (1.26) and (1.28), we obtain
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$(1.29)$ 
$(n-1)\varphi^{ht}p_{,t}=-0.$

Transvecting $(1.29)$ with $\varphi_{hk}$ and taking account of the fact that $\eta^{t}p_{,t}=-0$ we obtain

$(1.30)$ 
$(n-1)p_{,t}=-0.$

Thus we see that if $n>1$, then $p=\text{constant}$.

On the other hand, if we transvect $(1.11)$ with $\eta^{i}$, then we have $p_{,t}(v\eta^{t})=2n\rho$, where we have put $v=\nabla_{v}v'$. Then by the Green's theorem, we see that if $n>1$, then

$(1.31)$ 
$p=0.$

Next, we investigate on the case of $n=1$, that is, $2n+1=3$. It is well known that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold. Therefore we have the following formula in the case of $n=1$:

$(1.32)$ 
$K_{kij}^{i}+K_{kij}^{i}K_{ij}+g_{ki}K_{j}^{j}g_{j}^{j}-K_{kij}^{i}-K_{j}^{j}-\frac{K}{2}(g_{ki}g_{j}^{j}-g_{ji}g_{k}^{k})=0.$

Transvecting $(1.32)$ with $\eta_{i}^{j}$, we obtain

$(1.33)$ 
$K_{kij}^{i}=-\frac{K}{2}\gamma_{ki}$

by virtue of $(1.1)$.

Substituting $(1.33)$ into $(1.32)$, we obtain

$(1.34)$ 
$K_{kij}^{i}=\frac{K}{2}(\gamma_{k}^{h}\gamma_{j}^{j}-\gamma_{j}^{h}\gamma_{k}^{k}).$

Thus we have the following (Eum, [2])

**THEOREM 1.2.** A 3-dimensional cosymplectic manifold with constant scalar curvature $K$ is a manifold of constant curvature with respect to $\gamma_{ji}$.  

In the case of $n=1$, we obtain

$(1.35)$ 
$p_{,t}=K_{kt}\mathcal{L}_{v}\eta^{t}$

by virtue of $(1.15)$ and $(1.22)$.

Substituting $(1.33)$ into $(1.35)$ and taking account of the fact that $\eta_{i}\mathcal{L}_{v}\eta^{t}=0$, we see that

$(1.36)$ 
$p_{,t}=\frac{K}{2}g_{kt}\mathcal{L}_{v}\eta^{t}$. 

Substituting (0.4) into the well known formula:

\[ \mathcal{L}_v(\nabla_j \eta^k) - \nabla_j (\mathcal{L}_v \eta^k) = (\mathcal{L}_v \{ h_{ji} \}) \eta^i, \]

we easily see that

\[ -\nabla_j (\mathcal{L}_v \eta^k) = \rho \gamma_j^k. \]

and from which

\[ \nabla_i (\mathcal{L}_v \eta^i) = -2\rho \]

by virtue of the fact that \( n = 1. \)

If the scalar curvature \( K \) is non-zero constant, then operating \( \nabla_j \) to (1.36) and taking account of (1.37), we obtain

\[ \nabla_j \nabla_k \rho + \frac{K}{2} \rho \gamma_{jk} = 0, \]

and from which

\[ \nabla_i \nabla_i \rho + K \rho = 0. \]

Under the assumption that \( K \) is non-zero constant, if we take account of (1.15), (1.17) and (1.33), then we obtain

\[ \frac{K}{2} \mathcal{L}_v g_{ji} = \nabla_i \rho_j - 2\nabla_j \rho_i - \eta^i \{(\nabla_i \rho_j) \eta_j + (\nabla_j \rho_i) \eta_i\} + \rho \eta_i, \]

where \( \rho \) is defined by (1.25).

Substituting (1.41) into the identity:

\[ \mathcal{L}_v \{ h_{ji} \} = \frac{1}{2} g^{hk}(\nabla_j \mathcal{L}_v g_{ki} + \nabla_i \mathcal{L}_v g_{kj} - \nabla_k \mathcal{L}_v g_{ij}), \]

and transvect the result with \( \eta^i \), we obtain

\[ K \rho \gamma_{jk} = \eta^i \nabla_i \rho_j - 2\eta^i \nabla_i \rho_j \eta_j - \eta^i \{(\nabla_i \nabla_j \rho) \eta_j + \eta^i (\nabla_i \nabla_j \rho) \eta_j\}, \]

where we have used the relation

\[ \eta^i \nabla_i \rho = 0 \]

which is obtained from (1.39).

Transvecting (1.42) with \( g^{ij} \) and taking account of (1.25) and (1.43), we obtain

\[ 2K \rho = -\eta^i \nabla_i \rho. \]

Substituting \( \nabla_i \nabla_i \rho = \nabla_i \nabla_i \rho - K \rho \) into (1.44), we obtain

\[ 2K \rho = -\nabla_i (\eta^i \nabla_i \rho). \]
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Substituting (1.24) into (1.45), we obtain

\begin{equation}
3K\rho = -\nabla_i\rho^i\rho.
\end{equation}

Comparing (1.40) with (1.46), we obtain in the case of $n=1$ also

\begin{equation}
\rho = 0
\end{equation}

by virtue of the assumption $K \neq 0$.

Taking account of (1.31) and (1.47), we have the following

**Theorem 1.3.** If a compact cosymplectic manifold $M$ of dimension $2n+1$ $(n \geq 1)$ admits an $\eta$-projective vector field $v^i$ and the scalar curvature $K$ of $M$ is non-zero constant, then the associated vector $p^i$ of $v^i$ belongs to the distribution orthogonal to $\eta^i$, that is, $p_i\eta^i=0$.

2. Lie derivations with respect to an $\eta$-projective vector in a cosymplectic manifold

In the present section, we calculate the Lie derivations of some geometrical objects in the cosymplectic manifold $M$ admitting an $\eta$-projective vector field $v^i$.

Substituting (1.31) into (1.11), we obtain

\begin{equation}
\nabla_j v = (2n+1)p_j,
\end{equation}

where $v=\nabla_i v^i$. Thus $p_i$ is a gradient vector.

Substituting the fact that $\nabla_j p_i = \nabla_i p_j$ into (1.16), we obtain

\begin{equation}
\mathcal{L}_v K_{ji} = (\nabla_k p_i)\gamma_j^k - (\nabla_j p_i)\gamma_k^k.
\end{equation}

Substituting (1.23) and (1.31) into (1.19), we obtain

\begin{equation}
\eta^i \nabla_i p_j = 0.
\end{equation}

Thus we see from (1.17) that

\begin{equation}
\mathcal{L}_v K_{ji} = - (2n-1)\nabla_j p_i,
\end{equation}

by virtue of (1.31), (2.1) and (2.3).

Substituting (0.4) into the formula

\begin{equation}
\nabla_k \mathcal{L}_v \epsilon_{ji} = \nabla_k (\nabla_j v_i + \nabla_i v_j),
\end{equation}

we obtain

\begin{equation}
\nabla_k \mathcal{L}_v \epsilon_{ji} = 2p_k \gamma^i_{ji} + p_j \gamma^k_{ki} + p_i \gamma^i_{jk}
\end{equation}
We define a tensor field $G_{ji}$ on $M$ by

$$G_{ji} = K_{ji} - \frac{K}{2n} \gamma_{ji},$$

where $K$ is the scalar curvature of $M$, then we see easily that

$$G_{ji} = G_{ij}, \quad G_{ji} g^{ji} = G_i = 0, \quad \gamma^i G_{ji} = 0.$$  

Denoting the Lie derivation with respect to $\eta$ by $\mathcal{L}_\eta$ in $M$, we obtain

$$\mathcal{L}_\eta \{ h \} = \nabla_j \nabla_i \eta^h + \eta^i K_{jli}^h = 0$$

and from which

$$\mathcal{L}_\eta K_{jli}^h = \eta^i \nabla_t K_{jli}^h = 0.$$  

Contracting with respect to $h$ and $k$, we obtain

$$\eta^i \nabla_t K_{jli} = 0.$$  

We define an $\eta$–projective curvature tensor by

$$P_{kji}^h = K_{kji}^h - \frac{1}{2n-1} (\gamma_j^h K_{kii} - \gamma_j^h K_{kii}),$$

then this tensor field satisfies

$$P_{kji}^h = -P_{jki}^h, \quad P_{kji}^t = 0, \quad P_{tji}^t = 0, \quad P_{kji}^h + P_{ikj}^h + P_{jik}^h = 0$$

and

$$P_{kji}^h g^{ji} = \frac{2n}{2n-1} G_k^h, \quad P_{kji}^h \eta_h = 0.$$  

If the scalar curvature $K$ is non-zero constant, then using

$$\nabla^j K_{ji} = \frac{1}{2} \nabla_i K = 0, \quad \nabla^i K_{kji}^t = \nabla_k K_{jli}^i - \nabla_j K_{kli}$$

and

$$\eta^i \nabla_k K_{jli} = 0$$

which is obtained from (2.9), we see that

$$\nabla^k P_{kji}^h = \frac{2(n-1)}{2n-1} \nabla^h G_{ji} - \nabla_i G_j^h.$$
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Calculating \( \mathcal{L}_\gamma G_{ji} \) and taking account of (2.4) and (2.7), we can see that

\[
(2.15) \quad \mathcal{L}_\gamma G_{ji} = -(\nabla_j w_i + \nabla_i w_j)
\]

if the scalar curvature is non-zero constant, where we have put

\[
(2.16) \quad w^k = \frac{2n-1}{2} \eta^k + \frac{K}{2n} v^k.
\]

Substituting (2.2) and (2.4) into the Lie derivation of (2.10) and taking account of (1.15), we obtain

\[
(2.17) \quad \mathcal{L}_\gamma P_{kji} = \frac{1}{2n-1} \left\{ \left[ \mathcal{L}_\gamma (\eta^i v^h) \right] K_{ki} - \left[ \mathcal{L}_\gamma (\eta^i v^h) \right] K_{ji} \right\} = \frac{1}{2n-1} \left\{ \eta_i \eta^i (\eta^j v^h) K_{ki} - \eta_k \eta^j (\eta^i v^h) K_{ji} \right\}.
\]

We assume that the scalar curvature \( K \) is non-zero constant. In this case, we obtain

\[
(2.18) \quad (\eta^k \mathcal{L}_\gamma P_{kji}) g^{ji} = 0
\]

by virtue of (0.4) and (1.1).

From the first equation of (2.12), we obtain

\[
(\mathcal{L}_\gamma P_{kji}) g^{ji} + P_{kji} \mathcal{L}_\gamma g^{ji} = \frac{2n}{2n-1} \mathcal{L}_\gamma G^k
\]

and from which

\[
(2.19) \quad (\eta^k \mathcal{L}_\gamma P_{kji}) g^{ji} + (\eta^k P_{kji}) \mathcal{L}_\gamma g^{ji} + P_{kji} (\eta^k \mathcal{L}_\gamma g^{ji}) = \frac{2n}{2n-1} \eta^k \mathcal{L}_\gamma G^k.
\]

Substituting (2.6), (2.14) and (2.18) into (2.19), we obtain

\[
(2.20) \quad \left[ \frac{2n}{2n-1} \eta^k \mathcal{L}_\gamma G^k - \left\{ \frac{2(n-1)}{2n-1} \eta^h G_{ji} - \nabla_j G^h_{ji} \right\} \mathcal{L}_\gamma g^{ji} + \frac{2n}{2n-1} G^h_{ji} \right] w_k = 0.
\]

3. A decomposition of an \( \eta \)-projective vector in a compact cosymplectic manifold with non-zero constant scalar curvature

In the present section, we use for briefness the following notations:

\[
(3.1) \quad I_1 = \int_M G_{ji} h^j w^h dV, \quad I_2 = \int_M (\nabla_k G_{ji}) (\mathcal{L}_\gamma g^{ji}) w^k dV,
\]
$I_3 = \int_M (V^j G_{jk})(L_x g^{ji}) w^k dV$, $I_4 = \int_M (\nabla^k L_x G_k^h) w dV$,
where $dV$ denotes the volume element of $M$, and

(3.2) \[ \alpha = (V^i w^i)^2, \quad \beta = (V_j w_i + V^j w_i)(V^t w^i + V^i w^t). \]

Let $M$ be a compact cosymplectic manifold with non-zero constant scalar curvature and let $M$ admits an $\eta$–projective vector field $v^i$ defined by (0.4). In this case, we shall calculate the values of the integrals (3.1) following same ways as the processes of [3].

By using the identity

\[ V_j A^j = V^j V_j p_i - K t_{ji} p^i, \]

where $p_i = v^j p$ and $A = g^{ji} V_j v_i$, and taking account of (1.31) and (2.7), we obtain

\[ I_1 = -\int_M (V^i A^j p^j) w^i dV + \int_M (V^i V_j p_i) w^j dV - \frac{K}{2n} \int_M p_t w^i dV. \]

Taking account of (2.16) and the Green's theorem, we obtain

\[
\begin{align*}
-\int_M (V^i A^j p) w^i dV &= \int_M (A^j p^j) (V^i w^i) dV \\
&= \frac{2}{2n-1} \int_M \alpha \, dV - \frac{K}{n(2n-1)} \int_M (V_i v^i) (V^i w^i) dV \\
&= \frac{2}{2n-1} \int_M \alpha \, dV + \frac{K}{n(2n-1)} \int_M (V_i v^i) w^i dV \\
&= \frac{2}{2n-1} \int_M \alpha \, dV + \frac{(2n+1)K}{n(2n-1)} \int_M p_t w^i dV.
\end{align*}
\]

Consequently, we have

\[ I_1 = \frac{2}{2n-1} \int_M \alpha \, dV + \int_M (V^i V_j p_i) w^j dV + \frac{(2n+3)K}{n(2n-1)} \int_M p_t w^i dV. \]

By using

\[ V^i (V_i v_i + V^i v_j) = V^j L_x g^{ji} = (2n+3) p_i \]

we see that

\[
\begin{align*}
\int_M (V^i V_j p_i) w^j dV &+ \frac{(2n+3)K}{2n(2n-1)} \int_M p_v w^i dV \\
&= \frac{1}{2n-1} \left[ \int_M \left\{ V^j \left( \frac{2n-1}{2} V_i p_i + \frac{K}{2n} V_i v_i \right) \right\} w^i dV \\
&\quad + \int_M \left\{ V^i \left( \frac{2n-1}{2} V_j p_j + \frac{K}{2n} V_i v_j \right) \right\} w^i dV \right]
\end{align*}
\]
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\[
\begin{align*}
&= \frac{1}{2n-1} \int_M \mathcal{P}^j (\mathcal{P}_j \omega_i + \mathcal{P}_i \omega_j) \omega^i \text{d}V \\
&= \frac{1}{2n-1} \left[ \int_M \mathcal{P}^j ((\mathcal{P}_j \omega_i + \mathcal{P}_i \omega_j) \omega^i) \text{d}V \\
&\quad - \int_M (\mathcal{P}_j \omega_i + \mathcal{P}_i \omega_j) \mathcal{P}^j \omega^i \text{d}V \right] \\
&= -\frac{1}{2(2n-1)} \int_M \beta \text{d}V.
\end{align*}
\]

Substituting this equation into above equation, we obtain

\[
(3.3) \quad I_1 = -\frac{2}{2n-1} \int_M \alpha \text{d}V - \frac{1}{2(2n-1)} \int_M \beta \text{d}V.
\]

The integral $I_2$ is expressed by

\[
I_2 = \int_M \mathcal{P}_h [G_{ji} (\mathcal{L}_v \mathcal{L}_h \mathcal{G}^{ji}) \omega^h] \text{d}V - \int_M G_{ji} (\mathcal{P}_h \mathcal{L}_v \mathcal{G}^{ji}) \omega^h \text{d}V \\
- \int_M (G_{ji} \mathcal{L}_v \mathcal{G}^{ji}) \mathcal{P}_h \omega^h \text{d}V.
\]

Substituting (2.6) into this equation, we obtain

\[
I_2 = 2 \int_M G_{ji} \mathcal{P}_i \omega^i \text{d}V - \int_M \{\mathcal{L}_v (G_{ji} \mathcal{G}^{ji}) - g^{ji} \mathcal{L}_v G_{ji}\} \mathcal{P}_i \omega^i \text{d}V \\
= 2 I_1 - \int_M g^{ji} (\mathcal{P}_j \omega_i + \mathcal{P}_i \omega_j) \mathcal{P}_i \omega^i \text{d}V.
\]

Hence we get

\[
(3.4) \quad I_2 = 2 I_1 - 2 \int_M \alpha \text{d}V.
\]

Since

\[
(3.5) \quad g^{kj} \mathcal{L}_v (\mathcal{P}_k G_{ji}) = g^{kj} [\mathcal{P}_k \mathcal{L}_v G_{ji} - (\mathcal{L}_v \{t_{kj}\} G_{ji} - (\mathcal{L}_v \{t_{ki}\}) G_{ji}]] \\
= \mathcal{P}_j \mathcal{L}_v G_{ji} - 3 G_{ji} \mathcal{P}_i,
\]

and

\[
(3.6) \quad \int_M (\mathcal{P}_j \mathcal{L}_v G_{ji}) \omega^i \text{d}V = -\int_M [\mathcal{P}_j (\mathcal{P}_j \omega_i + \mathcal{P}_i \omega_j)] \omega^i \text{d}V = \frac{1}{2} \int_M \beta \text{d}V,
\]

which is obtained from (2.15), the integral $I_3$ is expressed by

\[
I_3 = \int_M \{\mathcal{L}_v [\mathcal{P}_j G_{ik}) \mathcal{G}^{ji}] \omega^h \text{d}V - \int_M g^{ji} (\mathcal{L}_v \mathcal{P}_j G_{ik}) \omega^h \text{d}V \\
= -\int_M (\mathcal{P}_j \mathcal{L}_v G_{ji}) \omega^i \text{d}V + 3 \int_M G_{ji} \mathcal{P}_i \omega^i \text{d}V.
\]
Hence substituting (3.6) into this equation, we obtain

\[(3.7) \quad I_3 = -\frac{1}{2} \int_M \beta dV + 3I_1.\]

Lastly, we calculate the integral \(I_4\).

\[
I_4 = \int_M \left[ \nabla^k \left( \mathcal{L}_v \left( G_{kj} g^{jk} \right) \right) \right] w_k dV
\]

\[
= \int_M \left( \nabla^k \mathcal{L}_v G_{kj} \right) w_j dV + \int_M \left[ \nabla^k \left( G_{kj} \mathcal{L}_v g^{jk} \right) \right] w_k dV.
\]

Substituting (3.6) into this equation and taking account of (2.6), we obtain

\[
I_4 = \frac{1}{2} \int_M \beta dV - 3 \int_M G_{ij} p^i w^j dV = \frac{1}{2} \int_M \beta dV - 3I_1.
\]

Thus we have

\[(3.8) \quad I_4 = -I_3.\]

Integrating (2.20) over \(M\), we obtain

\[(3.9) \quad 2n(I_1 + I_4) - 2(n-1)I_2 + (2n-1)I_3 = 0.\]

Substituting (3.3), (3.4), (3.7) and (3.8) into (3.9), we obtain

\[(3.10) \quad 2(2n-3) \int_M \alpha dV + \int_M \beta dV = 0.\]

In the case of \(n > 1\), since \(\alpha > 0\) and \(\beta > 0\) over \(M\) we find that

\[(3.11) \quad \alpha = 0, \quad \beta = 0.\]

That is, we obtain

\[(3.12) \quad \nabla_i w^i = 0, \quad \nabla_j w_i + \nabla_i w_j = 0.\]

Therefore \(w^k\) is a Killing vector.

We consider in the case of \(n = 1\). Since \(p_i\) is a gradient vector and \(K\) is non-zero constant, we obtain

\[(3.13) \quad \nabla_i \mathcal{L}_v p_i = 0\]

by virtue of (1.19), (1.20) and (1.47)

Substituting (1.47) and (3.13) into (1.41), we obtain

\[(3.14) \quad \frac{K}{2} \mathcal{L}_v e_{ji} = -\nabla_j p_i;\]

and from which
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\( (3.15) \)

\[
\frac{K}{2} (\mathcal{V}_j v_i + \mathcal{V}_i v_j) = -\mathcal{V}_j p_i.
\]

(3.15) and the fact that \( \mathcal{V}_j p_i = \mathcal{V}_i p_j \) shows that if we put

\[
w^h = \frac{1}{2} p^h + \frac{K}{2} v^h,
\]

then \( w^h \) is a Killing vector, that is,

\( (3.16) \)

\[
\mathcal{V}_j w_i + \mathcal{V}_i w_j = 0.
\]

Thus an \( \eta \)-projective vector \( v^h \) is decomposed in the form

\( (3.17) \)

\[
v^h = \frac{2n}{K} \left( w^h - \frac{2n-1}{2} p^h \right),
\]

where \( w^h \) is a Killing vector and \( p_i \) is a gradient vector.

The uniqueness of this decomposition is proved by the following way. In fact if

\[
w^h - \frac{2n-1}{2} p^h = w^h - \frac{2n-1}{2} p^h,
\]

then \( p^h - \nu p^h \) also a Killing vector. On the other hand, since \( p_i - \nu p_i \) is a gradient vector, we see that

\[
\mathcal{V}_j (p_i - \nu p_i) = 0,
\]

and from which

\[
\mathcal{V}_j \mathcal{V}_i (p - \nu p) = 0
\]

where we have put \( \mathcal{V}_i p = p_i, \mathcal{V}_i p = \nu p_i. \)

Since \( M \) is compact and orientable, we see that \( \nu p - \nu p \) is a constant. (Yano, [5]) Thus we obtain \( p_i = \nu p_i. \) Therefore the uniqueness of the decomposition is proved.

Substituting (2.16) into the second equation of (3.12), we obtain

\[
(2n-1) \mathcal{V}_j p_i + \frac{K}{2n} (\mathcal{V}_j v_i + \mathcal{V}_i v_j) = 0.
\]

Operating \( \mathcal{V}_k \) to this equation and taking account of (0.4), we obtain

\( (3.18) \)

\[
\mathcal{V}_k \mathcal{V}_i p_i + \frac{K}{2n(2n-1)} (2p_k \nu_{ji} + p_j \nu_{ki} + p_i \nu_{kj}) = 0.
\]

Transvecting (3.18) with \( g^{kj} \), we see that

\( (3.19) \)

\[
\mathcal{V}_i \mathcal{V}_i p^h = -\frac{(2n+3)K}{2n(2n-1)} p^h.
\]
Therefore $K > 0$ since $K$ is non-zero constant. (Yano, [5])

Taking account of (0.4) and (3.16), we easily obtain

$$(3.20) \quad \mathcal{L}_w \{h\}_{ji} = \frac{2n-1}{2} \mathcal{L}_p \{h\}_{ji} + \frac{K}{2n} \mathcal{L}_v \{h\}_{ji},$$

where $\mathcal{L}_w$ and $\mathcal{L}_p$ indicate the Lie derivations with respect to $w^h$ and $p^h$ respectively.

Since $w^h$ is a Killing vector, we see that

$$\mathcal{L}_p \{h\}_{ji} = -\frac{K}{n(2n-1)} \mathcal{L}_v \{h\}_{ji}$$

by virtue of the identity (III) of section 1 and (3.20). Thus we obtain

$$\mathcal{L}_p \{h\}_{ji} = -\frac{K}{n(2n-1)} (\pi_{ji}^h + \pi_{ij}^h).$$

Therefore, taking account of above results, we have the following

**Theorem 3.1.** Let $M$ be a compact $(2n+1)$ $(n \geq 1)$ dimensional cosymplectic manifold with non-zero constant scalar curvature $K$. If $M$ admits an $\eta$-projective vector field $v^h$ defined by (0.4), then $K > 0$ and $v^h$ is decomposed uniquely in the form:

$$v^h = \frac{2n}{K} \left( w^h - \frac{2n-1}{2} p^h \right),$$

where $w^h$ is a Killing vector field and $p^h$ is the associated vector field of $v^h$. Moreover $p^h$ is also an $\eta$-projective vector field and the associated vector field of $p^h$ is proportional (with constant coefficient $-\frac{K}{n(2n-1)}$) to $p^h$ itself.

Substituting (3.18) into the identity

$$V_k V_j p^h - V_j V_k p^h = K_{kji} p^i,$$

we obtain

$$(3.21) \quad U_{kji}^h p^i = 0,$$

where we have put

$$(3.22) \quad U_{kji}^h = K_{kji}^h - \frac{K}{2n(2n-1)} (\gamma_k^h \gamma_{ji} - \gamma_j^h \gamma_{ki}).$$

Since $w^h$ is a Killing vector field, we obtain

$$(3.23) \quad \mathcal{L}_v g_{ji} = (V_j v_i + V_i v_j) = -\frac{2n(2n-1)}{K} V_j p^i$$
by virtue of (2.16), and from which

\[ \mathcal{L}_{\omega}g^{ij} = \frac{2n(2n-1)}{K} \mathcal{V}^i p^j. \]

Substituting (2.24) into \( \mathcal{L}_{\omega}T^i = \mathcal{L}_{\omega}(g^{hi}T_i), \) we obtain

\[ \mathcal{L}_{\omega}T^i = 0 \]

by virtue of (1.15), (1.31) and (2.24).

From (3.25), we obtain

\[ \mathcal{L}_{\omega}T^i = 0. \]

Taking account of (1.15), (2.2), (3.22), (3.23) and (3.26), we see that

\[ \mathcal{L}_{\omega}U_{ki} = 0. \]

If we substitute (3.27) into the identity:

\[ \mathcal{L}_{\omega}\mathcal{V}iU_{ki} = \mathcal{V}i\mathcal{L}_{\omega}U_{ki} \]

\[ = U_{ki}i\mathcal{L}_{\omega}\left[ h \right] - U_{li}i\mathcal{L}_{\omega}\left[ k \right] - U_{li}i\mathcal{L}_{\omega}\left[ t \right] - U_{jki}i\mathcal{L}_{\omega}\left[ l \right] - U_{kij}i\mathcal{L}_{\omega}\left[ l \right], \]

then, we obtain

\[ \mathcal{L}_{\omega}\mathcal{V}iK_{kj} = -(2U_{kij}p_i + U_{lij}p_k + U_{ki}p_j + U_{kj}p_i) \]

by virtue of (3.21).

Transvecting this equation with \( p^k \) and taking account of (3.21), we obtain

\[ \mathcal{L}_{\omega}\mathcal{V}iK_{ki} = -U_{lij}p_k p^k. \]

Contracting with respect to \( k \) and \( i \) in (3.28), we obtain

\[ \mathcal{L}_{\omega}\mathcal{V}iK_{ji} = -G_{ij}p_i, \]

and from which

\[ \mathcal{L}_{\omega}\mathcal{V}iK_{ji} = -G_{ij}p_i. \]

Thus by (3.21), (3.22), (3.29) and (3.31), we have the following

THEOREM 3.2. Under the same assumption for \( M \) as the theorem 3.1, we have the following propositions.

If one of the following two conditions is satisfied, then \( M \) is a cosymplectic manifold of constant curvature with respect to \( \gamma_{ji}. \)

1. The Lie algebra of all \( \eta \)-projective vectors is transitive in \( M. \)
(2) $M$ is a symmetric manifold. Moreover if $M$ is a manifold of Ricci parallel, then $M$ is an $\eta$-Einstein manifold.

4. Hypersurfaces of a cosymplectic manifold admitting an $\eta$-projective vector field

We consider the distribution orthogonal to $\eta^h$ in a $(2n+1)$-dimensional cosymplectic manifold $M$.

If $X$ and $Y$ are vectors contained in such a distribution, then $[X, Y] = \nabla_X Y - \nabla_Y X$ is also contained in such a distribution. Therefore by a theorem of Frobenius, such a $2n$-dimensional distributions is integrable. Moreover, such a distribution is evidently parallel. Therefore $M$ is locally a product manifold of a $2n$-dimensional Riemannian manifold and a $1$-dimensional Riemannian manifold. If $M$ is complete and simply connected, then there exists a hypersurface $M^{2n}$ of $M$ such that

$$M = M^{2n} \times \mathbb{R}^1,$$

$\eta^h$ is normal to $M^{2n}$ and $M^{2n}$ is complete and simply connected.

Let the hypersurface $M^{2n}$ is covered by a system of coordinate neighborhoods $\{V; y^a\}$, then $M^{2n}$ is expresses by $x^h=x^h(y^a)$. Denoting $B^h_a = \partial_a x^h$, $(\partial_a = \partial/\partial y^a)$, the induced metric tensor $g_{ba}$ on $M^{2n}$ from that of $M$ is given by $g_{ba} = B^i_b B^j_a g_{ij}$.

Taking account of the fact that $\nabla_b = B^i_b \nabla_j$, $\nabla_b$ being the operator of covariant differentiation with respect to $g_{ba}$, and the Weingarten's formula: $\nabla_b \eta^h = -h^a_b B^h_a$, $h^a_b$ being the second fundamental tensor of $M^{2n}$, we easily see that $h^a_b = 0$, that is, $M^{2n}$ is a totally geodesic hypersurface of $M$. Therefore, the Gaussian equation for $M^{2n}$ is given by

$$K_{dcb^a} = K_{hkb} B^h_d B^k_c B^i_b B^j_a,$$

where $K_{dcb^a}$ is the curvature tensor of $M^{2n}$.

We denote by $(B^a_b, \eta^h)$ the inverse matrix of the matrix $(B^h_b)$. In this case, an $\eta$-projective vector field $v^h$ of $M$ is decomposed in the form

$$v^h = B^h_a u^a + \alpha \cdot \eta^h,$$

where $u^h = B^h_a v^a$ is a covector field of $M^{2n}$.

Taking account of the fact that $h_{ba} = 0$ and (4.2), we easily verify
the following equation

\[ B^h B^j B^a_h (\nabla_k \nabla_j \nu^b + \nu^a K_{b,k}) = \nabla_k \nu^a + u^a K_{e,c} \]

Substituting (4.4) into (4.4), we obtain

\[ \mathcal{L}_u \left[ \frac{1}{\kappa_k} \right] = \nabla_k \nu^a + u^a K_{e,c} = \delta \frac{a}{D} \frac{a}{D} \frac{a}{D} \frac{a}{D}, \]

where \( t_a = B^a_k \nu^k \) and \( \mathcal{L}_u \) denotes the Lie derivation with respect to \( u^a \) in \( M^{2n} \). Thus we have the following

**THEOREM 4.1.** Let \( M \) be a \((2n+1)\)-dimensional complete and simply connected cosymplectic manifold. Then \( M \) is a product manifold of a totally geodesic hypersurface \( M^{2n} \) and a 1-dimensional Riemannian manifold \( R^1 \). If \( M \) admits an \( \eta \)-projective vector field \( \nu^k \), then \( M^{2n} \) admits a projective vector field \( u^a \).

On the other hand, transvecting (3.18) with \( B^a_k B^j B^i_a \), we obtain

\[ \nabla_k \nu^a + \frac{K}{2n} (2g_{ba} t^b + g_{ca} t^b + g_{ca} t^a) = 0, \]

where \( K \) is the constant scalar curvature of \( M \) and \( t_a = \nabla_a \nu^a \).

Transvecting (4.2) with \( g^{ab} g^{bc} \), we easily verify the fact that the scalar curvature of \( M^{2n} \) is equal to the scalar curvature of \( M \). Taking account of a theorem of Obata ([4]) and (4.6), we obtain the following (cf. Theorem A of [3])

**THEOREM 4.2.** Let \( M \) be a \((2n+1)\)-dimensional compact, connected and simply connected cosymplectic manifold with non-zero constant scalar curvature \( K \). If \( M \) admits an \( \eta \)-projective vector field \( \nu^h \) then the hypersurface \( M^{2n} \) orthogonal to \( \nu^h \) is globally isometric to a sphere of radius \( \sqrt{2n(2n-1)/K} \) in the Euclidean \((2n+1)\)-space.

5. An \( \eta \)-projective vector field in a Sasakian manifold

If a set \((\varphi, \eta, g)\) of a tensor field \( \varphi \) of type \((1,1)\), a vector field \( \eta \) and a Riemannian metric tensor \( g \) satisfies (0.1), (0.2) and additionally

\[ \varphi_{ji} = \frac{1}{2} (\partial_j \eta_i - \partial_i \eta_j) \]

then, such a set is called a contact structure. A manifold with a normal contact structure is called a Sasakian manifold.

It is well known that in a Sasakian manifold, the following equations
are satisfied:

\begin{align}
(5.1) & \quad \nabla_j \eta^i = \varphi_i^i, \quad \nabla_j \varphi_i^i = -g_j \eta^i + \delta_j^i \eta_i, \\
(5.2) & \quad \eta_i K_{kji}^t = \eta_k g_{ji} - \eta_j g_{ki}, \\
(5.3) & \quad K_{jii}^t = 2 \alpha \eta_j.
\end{align}

In the present section, we investigate an $\eta$-projective vector field $v^h$ defined by

\begin{equation}
(5.4) \quad \mathcal{L}_v \left\{ \frac{h}{ji} \right\} = \nabla_j v^h + v^i K_{ji}^h = p_j \gamma_i^h + p_i \gamma_j^h
\end{equation}

in a Sasakian manifold.

Differentiating (5.4) covariantly, we obtain

\begin{equation}
(5.5) \quad \nabla_k \mathcal{L}_v \left\{ \frac{h}{ji} \right\} = (\nabla_k \gamma_i^h + (\nabla_k \gamma_j^h \gamma_j^h
\end{equation}

\begin{equation}
- p_j (\varphi_k \eta^h + \varphi_k \eta_i) - p_i (\varphi_k \eta^h + \varphi_k \eta_j).
\end{equation}

Substituting (5.5) into the identity:

\begin{equation}
\mathcal{L}_v K_{kji}^h = \nabla_k \mathcal{L}_v \left\{ \frac{h}{ji} \right\} - \nabla_j \mathcal{L}_v \left\{ \frac{h}{ki} \right\},
\end{equation}

we obtain

\begin{equation}
(5.6) \quad \mathcal{L}_v K_{kji}^h = (\nabla_k \gamma_i^h + (\nabla_k \gamma_j^h \gamma_j^h
\end{equation}

\begin{equation}
- p_j (\varphi_k \eta^h + \varphi_k \eta_i) + p_k (\varphi_j \eta^h + \varphi_j \eta_i)
\end{equation}

\begin{equation}
- p_i (2 \varphi_k \eta^h + \varphi_k \eta_j - \varphi_j \eta_k).
\end{equation}

Transvecting (5.6) with $\eta_i$, we find

\begin{equation}
(5.7) \quad \eta_i \mathcal{L}_v K_{kji}^h = p_k \varphi_{ji} - p_j \varphi_{ki} - 2 p_i \varphi_{kj}.
\end{equation}

Taking the Lie derivation of the both sides of (5.2), we obtain

\begin{equation}
\eta_i \mathcal{L}_v K_{kji}^t = (\mathcal{L}_v \eta_i) K_{kji}^t + (\mathcal{L}_v \eta_i) g_{ji} - (\mathcal{L}_v \eta_i) g_{ki} + \eta_k \mathcal{L}_v g_{ji} - \eta_j \mathcal{L}_v g_{ki}.
\end{equation}

Substituting (5.7) into this equation, we obtain

\begin{equation}
(5.8) \quad (\mathcal{L}_v \eta_i) K_{kji}^t = -p_k \varphi_{ji} + p_j \varphi_{ki} + 2 p_i \varphi_{kj} + (\mathcal{L}_v \eta_i) g_{ji} - (\mathcal{L}_v \eta_i) g_{ki} + \eta_k \mathcal{L}_v g_{ji} - \eta_j \mathcal{L}_v g_{ki}.
\end{equation}

Transvecting (5.8) with $\eta^h$ and taking account of (5.2), we obtain

\begin{equation}
(5.9) \quad \mathcal{L}_v g_{ji} = p_i \eta^h \varphi_{ji} + \eta_j \eta^h \mathcal{L}_v g_{ki}.
\end{equation}

Taking account of the symmetric property of $\mathcal{L}_v g_{ji}$ with respect to $j$ and $i$, we obtain
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(5.10) \[ 2p_i \eta^i \varphi_{ij} + \eta^k \left( \eta_j \mathcal{L}_v g_{ki} - \eta_i \mathcal{L}_v g_{kj} \right) = 0. \]

Transvecting (5.10) with $\eta^i$, we find

(5.11) \[ \eta^k \mathcal{L}_v g_{kj} = \nu \eta_j, \]

where we have put

(5.12) \[ \nu = \eta^k \eta^j \mathcal{L}_v g_{kj}. \]

Substituting (5.11) into (5.9), we see that

(5.13) \[ \mathcal{L}_v g_{ij} = \left( p_i \eta^i \right) \varphi_{ij} + V_1 \left( \eta^i \right)^j. \]

Transvecting (5.13) with $\varphi^{ji}$, we easily find

(5.14) \[ p_i \eta^i = 0 \]

and from which

(5.15) \[ \mathcal{L}_v g_{ij} = \nu \eta_i \eta_j \]

by virtue of (5.13).

Operating $\mathcal{P}_k$ to (5.15), we obtain

(5.16) \[ \mathcal{P}_k (\mathcal{P}_j v_i + \mathcal{P}_i v_j) = (\mathcal{P}_k \nu) \eta_i \eta_j + \nu (\varphi_{kj} \eta_i + \varphi_{ki} \eta_j). \]

Substituting (5.4) into (5.16) and transvecting the result with $\eta^j \eta^i$, we obtain $\mathcal{P}_k \nu = 0$, that is

(5.17) \[ \nu = \text{constant}. \]

On the other hand, substituting (5.14) into the identity:

\[ \mathcal{L}_v \{ h \}_{ji} = \frac{1}{2} g^{ht} (\mathcal{P}_j \mathcal{L}_v g_{ti} + \mathcal{P}_i \mathcal{L}_v g_{tj} - \mathcal{P}_i \mathcal{L}_v g_{ij} ), \]

and taking account of (5.17), we obtain

(5.18) \[ \mathcal{L}_v \{ h \}_{ji} = \nu (\varphi_j^i \eta_i + \varphi_i^j \eta_j). \]

Comparing (5.4) with (5.18), we obtain

(5.19) \[ p_i \eta^i + p_i \eta^h = \nu (\varphi_j^i \eta_i + \varphi_i^h \eta_j). \]

Transvecting (5.19) with $\eta_i$, we easily see that

\[ \nu = 0 \]

by virtue of (5.14), and from which

\[ p_i = 0, \quad \mathcal{L}_v \{ h \}_{ji} = 0. \]
Thus we have the following

**THEOREM 5.1.** In a Sasakian manifold, an \( \eta \)-projective vector field with an associated vector other than the zero vector does not exist.

**References**


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