CHARACTERIZATIONS OF ORDER IDEALS AND PERFECT SUBSPACES IN THE ORDERED NORMED SPACE OF $n \times n$ HERMITIAN MATRICES

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1. Preliminaries

In an ordered locally convex space, subspaces can be classified by various order properties. These order properties along with their interrelations are studied in detail in another paper by the author [1].

In this paper, we specifically consider the ordered normed space consisting of all the $n \times n$ Hermitian matrices, and give characterizations of order ideals and perfect subspaces. By these characterizations we get a concrete insight of what an order ideal or a perfect subspace should be and how they can be constructed.

We will denote $E$ to be the real normed space of $n \times n$ Hermitian matrices, where the norm of a matrix is taken to be the maximum of the absolute values of its eigenvalues. If $K$ is the set of matrices $A$ in $E$ whose eigenvalues are all nonnegative, i.e., $X^TAX \geq 0$ for all $n$-vector $X$, then $K$ is a generating cone in $E$. We take $K$ to be the positive cone of $E$.

An element $P \in V \cap K$, where $V$ is the closed unit ball of $E$, is called an extreme point of $V \cap K$ if whenever $P = \lambda Q + (1-\lambda) R$ for some $Q, R \in V \cap K, 0 < \lambda < 1$, we must have $P = Q = R$. It is easy to verify that if $P$ is an extreme point, then $0 \leq Q \leq P$ implies $Q = \lambda P$ for some $0 \leq \lambda \leq 1$.

We denote $e_k$ to be the unit vector whose all entries are zero except the $k$th one of value 1. We will denote $E_{kl}$ for $e_ke_k^T + e_le_l^T$ and $\bar{E}_{kl}$ for $i e_ke_k^T - i e_le_l^T$.

A subspace $J$ is called a perfect subspace or a nearly directed subspace [2] if for any $0 \leq a \in J, \varepsilon > 0$, there exists $b \in J$ such that $0, a \leq b + \varepsilon$. 

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where $e$ in this case is $e$ times the identity matrix. We will be using the fact that a subspace $J$ is a perfect subspace of $E$ if and only if $J^o$ is an order ideal in $E'$ [3].

2. A Characterization of extreme points

**Lemma 2.1.** Let $U$ be a unitary matrix, i.e. $\bar{U}^TU=1$. Then $A\succeq 0$ implies $\bar{U}^TAU\succeq 0$.

**Proof.** Let $X$ be an arbitrary $n$-vector. Then $\bar{X}^T(\bar{U}^TAU)X=(\bar{UX})^TA(\bar{UX})\succeq 0$ since $A\succeq 0$.

**Lemma 2.2.** Let $U$ be a unitary matrix and let $P\succeq 0$. Then $P$ is an extreme point of $V\cap K$ if and only if $\bar{U}^TPU$ is an extreme point of $V\cap K$.

**Proof.** Assume $P$ is an extreme point of $V\cap K$. By Lemma (2.1), $\bar{U}^TPU\in V\cap K$. Suppose $\bar{U}^TPU=\lambda Q+(1-\lambda)R$ for some $Q,R\in V\cap K$ and for $0<\lambda<1$. Then $P=\lambda UQ\bar{U}^T+(1-\lambda)UR\bar{U}^T$ with $UQ\bar{U}^T, UR\bar{U}^T\in V\cap K$. Hence $P=UQ\bar{U}^T=UR\bar{U}^T$ which in turn implies $\bar{U}^TPU=Q=R$. A similar proof for the if part is omitted.

**Lemma 2.3.** Let $D\succeq 0$ be a diagonal matrix. Then $D$ is an extreme point of $V\cap K$ if and only if $D=e_k e_k^T$ for some $1\leq k\leq n$.

**Proof.** First, we consider the if part. In view of Lemma (2.2), we may assume $D=e_k e_k^T$ without loss of generality. If $D=\lambda P+(1-\lambda)Q$ for some $P,Q\in V\cap K$ and $0<\lambda<1$, then $e_l^TDe_l=\lambda e_l^TPe_l+(1-\lambda)e_l^TQe_l$. Hence $0=\lambda p_{ll}+(1-\lambda)q_{ll}$ for $l\neq 1$. Since $p_{ll}\geq 0$, $q_{ll}\geq 0$, we must have $p_{ll}=q_{ll}=0$ for all $l$. Now, if $k\neq l$ and if $k$ and $l$ are different from 1, then from

$$(e_k+e_l)^TP(e_k+e_l)=p_{kl}+p_{lk}+\bar{p}_{kl}\geq 0$$

$$(e_k-e_l)^TP(e_k-e_l)=-p_{kl}-p_{lk}=-\bar{p}_{kl}\geq 0,$$

we must have $Re(p_{ll})=0$. Similarly, by applying $ie_k+ie_l$ we obtain $Im(p_{ll})=0$. Thus, we have

$$p_{lk}=p_{kl}=0 \quad k\geq l \quad l\neq k.$$

On the other hand, from $0\leq \lambda P\leq D$, we have

$$0\leq \lambda (e_1+ae_l)^TP(e_1+ae_l)\leq (e_1+ae_l)^Te_1e_1^T(e_1+ae_l)+l\neq 1$$

Hence, $\lambda (p_{1l}+\alpha p_{ll}+\alpha p_{ll})\leq 1 \quad \forall \alpha \in R$. Thus $Re(p_{1l})=0$. Similarly by applying $e_1+iae_l$, we obtain $Im(p_{1l})=0$. Therefore, we conclude
Next, we consider the only if part of the proof. Let \( D = (d_k) \) with \( d_l \neq 0 \) for some \( l \). Then clearly we have \( 0 \leq d_l e_l e_l^T \leq D \).

Since \( D \) is an extreme point, \( d_l e_l e_l^T = \lambda D \) for some \( 0 \leq \lambda \leq 1 \). Now, \( D = \alpha e_l e_l^T \) for some \( \alpha \geq 0 \) implies \( d_k = 0 \) \( \forall \) \( k \neq l \) and hence \( d_l = \lambda = 1 \).

**Theorem 2.4.** Let \( P \in V \cap K \). Then \( P \) is an extreme point of \( V \cap K \) if and only if \( P = U^T e_l e_k^T U \) for some unitary matrix \( U \) and for some \( 1 \leq k \leq n \).

**Proof.** If part follows from Lemma (2.3) and Lemma (2.2). For the only if part, recall that every Hermitian matrix is diagonalizable and hence \( P = U^T D U \) for some unitary matrix \( U \) and diagonal matrix \( D \). Now by Lemma (2.2), \( D \) is an extreme point. Hence the proof follows from Lemma (2.3).

**Corollary 2.5.** Let \( P \in V \cap K \). Then \( P \) is an extreme point if and only if \( P = X \bar{X}^T \) for some \( n \)-vector \( X \) with norm 1.

**Proof.** Only if part follows directly from Theorem (2.4) by setting \( X = U^T e_k \). For the if part, note that \( X \bar{X}^T \) is a matrix whose characteristic equation is of the form \( (\lambda - 1)\lambda^{n-1} \), and eigenvectors are \( X \) and \( (n-1) \) mutually orthogonal vectors each of which is orthogonal to \( X \). If we form a matrix \( U \) from these eigenvectors, \( X = U e_l \). Hence \( P = U e_l (U^T e_l)^T = U e_l e_l^T U \). Now, the proof follows from Theorem (2.4).

**Corollary 2.6.** Let \( P \in V \cap K \). Then \( P \) is an extreme point of \( V \cap K \) if and only if the characteristic equation of \( P \) has a single root of 1 and the other \( (n-1) \) roots are all zeros.

### 3. A characterization of order ideals

**Lemma 3.1.** Let \( J \) be a positively generated order ideal of \( E \). Then there exists \( Q \in J \cap K \) such that \( J \) is generated by \( Q \), i.e., \( J \) is the smallest order ideal containing \( Q \).

**Proof.** Let \( \mathcal{J} = \{ J_P | J_P \text{ is the order ideal generated by } P \in J \cap K \} \). Then \( \mathcal{J} \) is partially ordered by set inclusion. Since \( E \) is a finite dimensional vector space, every totally ordered subset of \( \mathcal{J} \) is finite and has a supremum. Thus, by Zorn's Lemma, there is a maximal
element $J_\circ$. Let $P_\circ$ be the generator of $J_\circ$, then clearly $J_\circ \subseteq J$.

Suppose $J_\circ \neq J$, then there exists $P \in J \cap K$ with $P \notin J_\circ$. But then the order ideal generated by $P_\circ + P$ contains $J_\circ$, which is a contradiction to the maximality of $J_\circ$.

**Lemma 3.2.** Let $I_m$ be the diagonal matrix whose first $m$ diagonal elements are 1 and others are 0, and let $J$ be the order ideal generated by $I_m$. Then $J = E_m$.

**Proof.** Since $E_m$ is a positively generated order ideal containing $I_m$, it is clear that $J \subseteq E_m$. To show the converse, let $A \in E_m$. We can write $A = \overline{U}^T D U$ for a diagonal $D \in E_m$ and a unitary matrix $U$ of the form

$$U = \begin{pmatrix} U_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

where $U_m$ is unitary in $E_m$ and $I_{n-m}$ is the identity matrix of order $n-m$.

Let $D^+$ be the diagonal matrix whose entries are the same as the corresponding entries in $D$ if they are positive and zero otherwise. Then

$$0, D \leq D^+ \leq \| A \| I_m$$

And similarly we have

$$0, -D \leq (-D)^+ = D^- \leq \| A \| I_m.$$ 

Therefore, $-\| A \| I_m \leq D \leq \| A \| I_m$ which implies

$$-\| A \| I_m \leq \overline{U}^T D U \leq \| A \| I_m,$$

and hence $A \in J$.

**Lemma 3.3.** Let $P \in K$ and let $J$ be the order ideal generated by $P$. Then there exists a unitary matrix $U$ and $m \leq n$ such that $J = \overline{U}^T E_m U$.

**Proof.** Let $P = \overline{U}^T D U$ where $D$ is diagonal and $U$ is a unitary matrix. Without loss of generality, we may assume that the diagonal elements $d_k$ of $D$ satisfies $d_k = 0 \forall k > m$ and $d_k \neq 0 \forall k \leq m$.

By (3.2), the order ideal generated by $D$ is $E_m$ since $\lambda I_m \leq D$ for $\lambda = \min \{ d_k | K = 1, 2, \ldots m \} \geq 0$. Therefore, the order ideal generated by $P = \overline{U}^T D U$ is $\overline{U}^T E_m U$.

**Theorem 3.4.** Let $J$ be a positively generated subspace. Then $J$ is an order ideal if and only if there exists a unitary matrix $U$ and $m \leq n$ such that $J = \overline{U}^T E_m U$. 

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Proof. If part of the Theorem is clear, For the only if part, use Lemma (3.1) to pick a generator $P$ of $J$. Then the theorem follows from Lemma (3.3).

Lemma 3.5. Let $H_m = \{ A \in E | a_{ij} = 0 \; \forall \; i, j \leq m, \; a_{ii} = 0 \; \forall \; i \geq m+1 \}$, and let $L$ be a subspace of $H_m$. If $J = E_m + L$, then $E_m = J \cap K - J \cap K$, where $E_m$ is the subspace of all $m \times m$ Hermitian matrices considered as an imbedded subspace of $E$.

Proof. If is clear that $E_m \subseteq J \cap K - J \cap K$. Hence, it is left to show $J \cap K \subseteq E_m$. Take an arbitrary element $A$ of $J \cap K$, then $A = B + C$ for some $B \in E_m$ and $C \subseteq L$. If $k \geq m+1$,

\[
(e_k + \bar{e}_l)^T A(e_k + \bar{e}_l) = a_{kk} + \bar{\lambda}a_{kl} + a_{kl} + \bar{\lambda}a_{ll} = b_{kk} + \bar{\lambda}b_{kl} + a_{kk} + \bar{\lambda}c_{kl} + \lambda c_{kl} + \bar{\lambda}c_{kl} = \lambda b_{kk} + \bar{\lambda}c_{kl} + \bar{\lambda}c_{kl} \geq 0 \; \forall \; \text{complex number } \lambda.
\]

Now, by choosing suitable value for $\lambda$, we get $c_{kl} = 0 \; \forall \; k \geq m+1$. Hence $C = 0$, which implies $A = B \in E_m$.

Corollary 3.6. If $H_m$ and $L$ are as in Lemma (3.5), and if $J = E_m + L$, then $J$ is an order ideal.

Proof. $J$ is an order ideal if and only if $J \cap K - J \cap K$ is an order ideal. But the latter is the same subspace as $E_m$ by Lemma (3.5), and $E_m$ is a positively generated order ideal in $E$. Therefore $J$ is an order ideal.

Lemma 3.7. If $J$ is an order ideal such that $J \cap K - J \cap K = E_m$ for some $m < n$, then $J = E_m + L$ where $L$ is a subspace of $H_m$ defined in Lemma (3.5).

Proof. First, we prove the case when $m = n - 1$. Enough to prove that for an arbitrary element $A \in J$, $a_{nn} = 0$. Assume that $a_{nn} \neq 0$. Without loss of generality, We may assume that $a_{nn} > 0$, $|a_{ii}| \leq 1$, $i = 1, 2, ..., n$. Let

\[
B = \begin{pmatrix}
\lambda & 0 & a_{1n} \\
0 & \lambda & a_{2n} \\
\lambda & \bar{a}_{nn} & a_{nn}
\end{pmatrix}
\]
where $\lambda$ is chosen such that $\lambda > \frac{16n}{a_{nn}}$. Then it is clear that $B \in J$ and $B \in E_m$.
We now prove that $B \geq 0$, i.e., $B \in K$ which would contradict the hypothesis $J \cap K = J \cap K = E_m$.

Let $X$ be an arbitrary $n$-vector such that $\sum_{i=1}^{n} |x_i|^2 = 1$.

Then
\[ X^t B X = \lambda \sum_{i=1}^{n} |x_i|^2 + \left( \sum_{i=1}^{n} \xi_i a_{nn} \right) x_n + \left( \sum_{i=1}^{n} a_{in} x_i \right) x_n \]

(1) when $|x_n| \geq \sqrt{1 - \left( \frac{a_{nn}}{4n} \right)^2}$

Since $\sum_{i=1}^{n} |x_i|^2 = 1$, we have
\[ |x_i| \leq \sqrt{1 - |x_n|^2} \leq \sqrt{1 - 1 + \left( \frac{a_{nn}}{4n} \right)^2} = \frac{a_{nn}}{4n}, \quad i = 1, 2, \ldots, n-1, \]

Thus,
\[ \sum_{i=1}^{n-1} |x_i| \leq \frac{a_{nn}}{4} \quad \text{and} \quad 2 \left( \sum_{i=1}^{n-1} |x_i| \right) |x_n| \leq a_{nn} |x_n|^2 \quad \text{from} \]

\[ |x_n| \geq \sqrt{1 - \left( \frac{1}{4n} \right)^2} \geq \frac{1}{2}. \]

Therefore,
\[ X^t B X \geq a_{nn} |x_n|^2 + \lambda (1 - |x_n|^2) - 2 \left( \sum_{i=1}^{n-1} |x_i| \right) |x_n| \geq 0 \]

(2) When $|x_n| \leq \sqrt{1 - \left( \frac{a_{nn}}{4n} \right)^2}$

From
\[ |x_n|^2 \leq 1 - \left( \frac{a_{nn}}{4n} \right)^2, \quad \text{we have} \quad \left( \frac{a_{nn}}{4n} \right)^2 \leq 1 - |x_n|^2 \]

and hence
\[ 1 \leq \left( \frac{4n}{a_{nn}} \right)^2 (1 - |x_n|^2). \]

By multiplying both sides by $1 - |x_n|^2$,
\[ 1 - |x_n|^2 \leq \left( \frac{4n}{a_{nn}} \right)^2 (1 - |x_n|^2)^2 \]
and hence,
\[ \sqrt{1 - |x_n|^2} \leq \frac{4n}{a_{nn}}(1 - |x_n|^2). \]

Now, since
\[ \sum_{k=1}^{l} |x_k| \leq 2\sqrt{\sum_{i=1}^{l} |x_i|^2} \quad \text{for all } l=1,2,\ldots, \]
We have
\[ 2\left(\sum_{i=1}^{n-1} |x_i|\right) |x_n| \leq 4\left(\sqrt{\sum_{i=1}^{n-1} |x_i|^2}\right) |x_n| \]
\[ = 4\sqrt{1 - |x_n|^2} \cdot |x_n| \]
\[ \leq \frac{16n}{a_{nn}} \left(1 - |x_n|^2\right) |x_n| \leq \lambda \left(1 - |x_n|^2\right) \]
Therefore,
\[ \tilde{X}^T B X = \lambda \sum_{i=1}^{n-1} |x_i|^2 + \sum_{i=1}^{n-1} (\bar{x}_i a_{in}) x_n + \left(\sum_{i=1}^{n-1} a_{in} x_i\right) \bar{x}_n + a_{nn} |x_n|^2 \]
\[ \geq \lambda \left(1 - |x_n|^2\right) - 2\left(\sum_{i=1}^{n-1} |x_i|\right) |x_n| + a_{nn} |x_n|^2 \]
\[ \geq \lambda \left(1 - |x_n|^2\right) - 2\left(\sum_{i=1}^{n-1} |x_i|\right) |x_n| \geq 0. \]

Next, we consider the case when \( m < n - 1 \). Note that \( J_{m+1} = J \cap E_{m+1} \) is an order ideal in \( E_n \) since both \( J \) and \( E_{m+1} \) are order ideals in \( E \). Hence for any element \( A \in J \), \( a_{m+1,m+1} = 0 \). Now, by mathematical induction \( a_{k,k} = 0 \) \( \forall \ k \geq m+1 \).

**Theorem 3.8.** Let \( J \) be an order ideal in \( E \). Then there exists a unitary matrix \( U \) such that \( J = \tilde{U}^T(E_m + L)U \) for some \( m \leq n \), where \( L \) is a subspace of \( H_m \).

**Proof.** Let \( J_1 = J \cap K - J \cap K \). Then \( J_1 \) is a positively generated order ideal. Hence by Theorem (3.4), there exists a unitary matrix \( U \) such that
\[ J_1 = \tilde{U}^T E_m U \quad \text{for some } m \leq n \]
Now, \( E_m = UJ_1 \tilde{U}^T \). If \( J_2 = UJ_1 \tilde{U}^T \) then \( J_2 \) is an order ideal since so is \( J \). Furthermore, \( J_2 \cap K - J_2 \cap K = E_m \) and hence by Lemma (3.7), \( J_2 \) is of the form \( E_m + L \) where \( L \) is a subspace of \( H_m \).

**Corollary 3.9.** Let \( J \) be a subspace of \( E \). Then \( J \) is an order ideal if and only if there exists a unitary matrix \( U \) such that \( J = \tilde{U}^T(E_m + L) \)
U for some $m \leq n$ and a subspace $L$ of $H_{m_0}$.

4. A characterization of perfect subspaces

**Proposition 4.1.** Let $f$ be a real linear functional on $E$. Then there exists $G \in E$ such that $f(A) = \frac{1}{2} \text{Tr}(GA + AG)$ $\forall A \in E$

**Proof.** Let $E_{kl} = e_k e_l^T + e_l e_k^T$ 
$\hat{E}_{k,l} = i e_k e_l^T - i e_l e_k^T$

$f_{k,l} = \frac{1}{2} f(E_{k,l}), \quad \hat{f}_{k,l} = \frac{1}{2} f(\hat{E}_{k,l})$

and define $G$ to be an $n \times n$ matrix whose $(k, l)$-element is $f_{k,l} + i \hat{f}_{k,l}$. Then it is easy to check that $G$ is Hermitian. Now, if we let $A_{k,l} = a_{k,l} + i \hat{a}_{k,l}$ with $a_{k,l}$ and $\hat{a}_{k,l}$ real then

$$f(A) = f \left( \sum_{k=1}^{n} \frac{1}{2} a_{kk} E_{kk} + \sum_{k,l=1}^{n} a_{k,l} E_{kl} + \sum_{k,l=1}^{n} \hat{a}_{k,l} \hat{E}_{k,l} \right)$$

$$= \sum_{k=1}^{n} a_{kk} f_{kk} + 2 \sum_{k,l=1}^{n} a_{k,l} f_{kl} + 2 \sum_{k,l=1}^{n} \hat{a}_{k,l} \hat{f}_{kl}$$

$$= \sum_{k=1}^{n} a_{k,l} f_{kl} + \sum_{k,l=1}^{n} \hat{a}_{k,l} \hat{f}_{kl}, \quad \text{where} \quad \hat{a}_{kk} = 0$$

and

$$(GA)_{kk} = \sum_{l=1}^{n} \left( f_{kl} + i \hat{f}_{kl} \right) (a_{lk} + i \hat{a}_{lk})$$

$$= \sum_{l=1}^{n} \left( f_{kl} a_{lk} - \hat{f}_{kl} \hat{a}_{lk} \right) + i \sum_{l=1}^{n} \left( f_{kl} \hat{a}_{lk} + \hat{f}_{kl} a_{lk} \right)$$

$$(AG)_{kk} = \sum_{l=1}^{n} \left( a_{kl} + i \hat{a}_{kl} \right) (f_{lk} + i \hat{f}_{lk})$$

$$= \sum_{l=1}^{n} \left( a_{kl} f_{lk} - \hat{a}_{kl} \hat{f}_{lk} \right) + i \sum_{l=1}^{n} \left( a_{kl} \hat{f}_{lk} + \hat{a}_{kl} f_{lk} \right)$$

$$= \sum_{l=1}^{n} \left( f_{kl} a_{lk} - \hat{f}_{kl} \hat{a}_{lk} \right) - i \sum_{l=1}^{n} \left( f_{kl} \hat{a}_{lk} + \hat{f}_{kl} a_{lk} \right)$$

Hence $(GA)_{kk} + (AG)_{kk} = 2 \sum_{l=1}^{n} \left( f_{kl} a_{lk} - \hat{f}_{kl} \hat{a}_{lk} \right)$

$$\frac{1}{2} \text{Tr}(GA + AG) = \sum_{k=1}^{n} \sum_{l=1}^{n} \left( f_{kl} a_{lk} - \hat{f}_{kl} \hat{a}_{lk} \right)$$

$$= \sum_{k,l} \left( f_{kl} a_{lk} + \hat{f}_{kl} \hat{a}_{lk} \right) = f(A)$$

**Lemma 4.2.** (a) $\text{Tr}(A E_{\alpha \beta} + E_{\alpha \beta} A) = 4 a_{\alpha \beta}$

(b) $\text{Tr}(A \hat{E}_{\alpha \beta} + \hat{E}_{\alpha \beta} A) = 4 \hat{a}_{\alpha \beta}$, where $A_{\alpha \beta} = a_{\alpha \beta} + i \hat{a}_{\alpha \beta}$

**Proof.** A routine proof of this is omitted.
LEMMA 4.3. Let $H_m = \{ A \in E | A_{ij} = 0 \quad \forall \quad i, j \leq m \}$. Then $H_m = E_m^o$.

Proof. To prove $H_m \subseteq E_m^o$, let $F \in H_m$ be an arbitrary element, then $F_{ij} = 0 \quad \forall \quad i, j \leq m$. Now, let $A \in E_m$ then $A_{ij} = A_{ij} = 0 \quad \forall \quad i \geq m + 1$. Hence $F_{kj} A_{jk} = 0 \quad \forall \quad j, k$ since if $k \geq m + 1$ then $A_{jk} = 0 \quad \forall \quad j = 1, 2, ..., m$ and if $k \leq m$ then $F_{kj} = 0$ for $j \leq m$ and $A_{jk} = 0$ for $j \geq m + 1$. Thus,

$$(FA)_{kk} = \sum_{j=1}^{n} F_{kj} A_{jk} = 0$$

and similarly,

$$(AF)_{kk} = \sum_{j=1}^{n} A_{kj} F_{jk} = 0$$

Therefore,

$$Tr(AF + FA) = 0.$$ 

To prove the converse, i.e., $E_m^o \subseteq H_m$, let $F \in E_m^o$. It is desired to show $F_{ij} = 0 \quad \forall \quad i, j \leq m$. But if $i, j \leq m$ then $E_{ij} \in E_m$ and $\hat{E}_{ij} \in E_m$. Also, $Tr(FE_{ij} + E_{ij} F) = 4f_{ij}$ and $Tr(F\hat{E}_{ij} + \hat{E}_{ij} F) = 4\hat{f}_{ij}$, by Lemma (4.2).

But these must be zeros since $F \in E_m^o$.

LEMMA 4.4. Let $L$ be a subspace of $E$ and let $U$ be a unitary matrix. Then $(U^TL^oU)^o = U^TL^oU$.

Proof. Let $F \in (U^TL^oU)^o$ then $Tr(FU^T A^o U + U^T A^o U F) = 0$, $\forall \quad A \in L$. Hence $Tr(U^T(UF U^T A) U + U^T(A^o U F U^T) U) = Tr(UF U^T A + A^o U F U^T) = 0$. Therefore, $UF U^T \in L^o$ and so $F \in U^TL^oU$.

Conversely, let $F \in U^TL^oU$ then $UF U^T \in L^o$. Hence by a similar computation as above, $F \in (U^TL)^o$.

LEMMA 4.5. Let $L$ be a perfect subspace of $E$ and $U$ be a unitary matrix. Then $U^TL^oU$ is a perfect subspace.

The trivial proof of this is omitted.

LEMMA 4.6. Given $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ such that

$$\sum_{k=1}^{n} |x_k| \leq \lambda_\epsilon \sum_{k=1}^{n} |x_k|^2 + \epsilon \quad \text{for all} \quad (x_1, x_2, ..., x_m) \in C^m \text{ with } \sum_{k=1}^{n} |x_k| \leq 1.$$ 

Proof. Choose $\lambda_\epsilon > \frac{m^2}{\epsilon^2}$. Then for $|x_k| < \frac{\epsilon}{m}$, $k = 1, 2, ..., m, \sum_{k=1}^{n} |x_k| < \epsilon$. Hence $\sum_{k=1}^{n} |x_k| < \epsilon + \lambda_\epsilon \sum_{k=1}^{n} |x_k|^2$. If $|x_l| \geq \frac{\epsilon}{m}$ for some $l$, then
LEMMA 4.7. Let $H_m = \{ A \in E | A_{ij} = 0 \forall i, j \leq m \}$, $m < n$ and $L$ be a subspace of $H_m$ such that $E_{kk} \in L \forall k > m$. Then $L$ is a perfect subspace.

Proof. Let $A \in L$ and let $M = \max \{|a_{ij}| |i \geq m + 1, j = 1, 2, \ldots, n\}$. Choose $N$ such that $N > M(m+n)\frac{(n-m)^2}{\varepsilon^2}$ and define a diagonal matrix $B$ such that $B = N \sum_{k=m+1}^{n} E_{kk}$. We claim that $A$, $0 \leq B + \varepsilon$.

To prove the claim, take an arbitrary vector $X$ with $\sum_{i=1}^{n} |x_i| = 1$. Then

$$\bar{X}^TAX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_j = \sum_{i,m+1} \sum_{j=1}^{n} a_{ij}x_j + \sum_{i=1}^{n} \sum_{j=m+1}^{n} a_{ij}x_j,$$

and hence

$$|\bar{X}^TAX| \leq \sum_{i,j} |a_{ij}| |x_i| |x_j| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}| |x_i| |x_j| + \sum_{i=m+1}^{n} \sum_{j=1}^{n} |a_{ij}| |x_i| |x_j|$$

$$\leq M \sum_{i=1}^{n} \sum_{j=m+1}^{n} |x_i| |x_j| + M \sum_{i=m+1}^{n} \sum_{j=1}^{n} |x_i| |x_j|$$

$$\leq M \sum_{i=1}^{n} \sum_{j=m+1}^{n} |x_j| + M \sum_{i=m+1}^{n} \sum_{j=1}^{n} |x_i| \leq M(m+n) \sum_{k=m+1}^{n} |x_k|$$

Now, by Lemma (4.6),

$$\sum_{k=m+1}^{n} |x_k| \leq \frac{(n-m)^2}{\varepsilon^2} \sum_{k=m+1}^{n} |x_k|^2 + \varepsilon$$

Therefore, $|\bar{X}^TAX| \leq M(n+m)$.

$$(n-m)^2 \sum_{k=m+1}^{n} |x_k|^2 + \varepsilon < N \sum_{k=m+1}^{n} \left|X_k\right|^2 + \varepsilon$$

$$= \bar{X}^T B X + \varepsilon$$

THEOREM 4.8. Let $L$ be a subspace of $E$. Then $L$ is a perfect subspace if and only if there exists $m \leq n$ such that $L = U^T L_1 U$ for some unitary $U$ and a subspace $L_1$ of $H_m$ with $E_{kk} \in L_1 \forall k \geq m$.

Proof. If $L$ is not a proper subspace, there is nothing to prove. Hence, we assume $L$ is a proper subspace. If part of the theorem follows from Lemma (4.6) and (4.7). To prove the only if part, note that since $L^o$ is an order ideal $L^o = U^T(E_m + L_2) U$ for some unitary matrix $U$ and a subspace $L_2$ of $H_{mo}$ by Theorem (3.8). Thus,
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\[ E_m + L_2 = UL^o U^T \]

and hence

\[ (E_m + L_2)^o = (UL^o U^T)^o = U L^o U^T = U L \]

by Lemma (4.4), and by the fact that \( L \) is a finite dimensional subspace which is always closed. From this, we obtain

\[ L = U^T (E_m + L_2)^o U \]

Now, let \( L_1 = (E_m + L_2)^o \). Then clearly, \( L_1 \) is a subspace of \( H_m \) since \( E_m \) is. It is left to show \( E_{kk} \in L_1 \) \( \forall k > m \). But, by Lemma (4.2), \( Tr(E_{kk} A + A E_{kk}) = 4 A_{kk} \) and if \( A \in E_m + L_2 \) with \( L_2 \subseteq H_m \) then \( a_{kk} = 0 \) \( \forall k \geq m + 1 \). Thus, \( E_k \in L_1 \).

References


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