CURVATURE TENSORS OF 3-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLDS

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1. 3-dimensional almost contact metric manifold

A \((2n+1)\)-dimensional differentiable manifold \(M\) is called to have an almost contact metric structure if there is given a positive definite Riemannian metric \(g_{ij}\) and a triplet \((\varphi_j^i, \xi^j, \eta_k)\) of \((1,1)\)-type tensor field \(\varphi_j^i\), vector field \(\xi^j\) and 1-form \(\eta_k\) in \(M\) which satisfy the following equations:

\[
\begin{align*}
\varphi_j^i \varphi_i^k &= -\delta_j^k + \eta_j \xi^k, \\
\varphi_j^i \xi^j &= 0, \\
\eta_i \varphi_j^i &= 0, \\
\eta_i \xi^i &= 1
\end{align*}
\]

and

\[
\begin{align*}
\xi \varphi_j^i \varphi_i^k &= g_{ij} - \eta_j \eta_i, \\
\eta_i &= g_{ii} \xi^i.
\end{align*}
\]

In this case, \(M\) is called a \((2n+1)\)-dimensional almost contact metric manifold. By virtue of the last equation of (1.2), we shall write \(\eta^h\) instead of \(\xi^h\).

We consider a \((0,4)\)-type tensor \(E_{kijh}\) in a \((2n+1)\)-dimensional almost contact metric manifold defined by

\[
E_{kijh} = (2n+1)(\gamma_{kijh} - \gamma_{jkih}) - \varphi_{ki} \varphi_{jh} + \varphi_{ji} \varphi_{kh} - 2\varphi_{kj} \varphi_{ih},
\]

where we have put

\[
\gamma_{ki} = g_{ki} - \eta_k \eta_i.
\]

By a direct computation, we obtain

\[
E_{kijh} E^{kijh} = 16(2n+1)n(n^2 - 1).
\]

Taking account of (1.3) and (1.5), we see that a \((0,4)\)-type tensor \(E_{kijh}\) in a 3-dimensional almost contact metric manifold defined by

\[
E_{kijh} = 3(\gamma_{kijh} - \gamma_{jkih}) - \varphi_{ki} \varphi_{jh} + \varphi_{ji} \varphi_{kh} - 2\varphi_{kj} \varphi_{ih}
\]

is a zero tensor.

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Therefore we obtain in a 3-dimensional almost contact metric manifold \( M \) the following identity:

\[
(1.7) \quad 3(\gamma_{ki} \gamma_{jh} - \gamma_{ji} \gamma_{kh}) = \phi_{ki} \phi_{jh} - \phi_{ji} \phi_{kh} + 2\phi_{kj} \phi_{ih},
\]
or equivalently

\[
(1.8) \quad 3(g_{ki} g_{jh} - g_{ji} g_{kh}) + 3\eta_k (g_{ji} \eta_h - g_{jh} \eta_i) - 3\eta_j (g_{ki} \eta_h - g_{kh} \eta_i) = \phi_{ki} \phi_{jh} - \phi_{ji} \phi_{kh} + 2\phi_{kj} \phi_{ih}.
\]

Thus we have the following

**Theorem 1.1.** In a 3-dimensional almost contact metric manifold, the identity (1.7) or equivalently (1.8) is satisfied.

On the other hand, it is well known fact that the conformal curvature tensor of Weyl vanishes identically in a 3-dimensional Riemannian manifold, that is, the following equation is satisfied:

\[
(1.9) \quad K_{kjih} + K_{ki} \delta_{jh} - K_{ji} \delta_{kh} + g_{ki} K_{jh} - g_{ji} K_{kh} + \frac{1}{2} (g_{ki} \delta_{jh} - g_{ji} \delta_{kh}) = 0,
\]

where \( K_{kjih}, \ K_{ji} \) and \( K \) are the curvature tensor, the Ricci tensor and the scalar curvature of the manifold respectively.

We call the section determined by a unit vector \( \nu^h \) orthogonal to \( \eta^h \) and the vector \( \partial^h \) a \( \phi \)-holomorphic section in a 3-dimensional almost contact metric manifold.

**2. Curvature tensor of 3-dimensional cosymplectic manifold**

If an almost contact metric structure of \( M \) introduced in the last section satisfies

\[
N_{ji}^h + (\partial_j \eta_i - \partial_i \eta_j) \eta^h = 0,
\]

where \( \partial_j = \partial / \partial x^j \), \( \{U, x^h\} \) being the coordinate neighborhoods and \( N_{ji}^h \) is the Nijenhuis tensor formed with \( \phi_j^h \), then \( M \) is called a normal almost contact metric manifold.

A normal almost contact metric manifold \( M \) is said to be cosymplectic if the 2-form \( \phi_j^h = \phi_j^h g_{ti} \) and the 1-form \( \eta_i \) are both closed. It is well known (Blair, [1]) that the cosymplectic structure of \( M \) is characterized by

\[
(2.1) \quad \nabla_j \phi_j^h = 0, \quad \nabla_j \eta_j = 0,
\]
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where $\nabla_k$ is the operator of covariant differentiation with respect to $g_{ji}$.

The following equations are also satisfied in cosymplectic manifold $M$:

\[(2.2) \quad K_{ji} \eta_i = 0, \quad K_{ji} \eta^i = 0\]

and

\[(2.3) \quad \eta^i \nabla_i K_{ji} = 0, \quad \eta^i \nabla_i K = 0.\]

In this section, we study on the curvature tensor of a 3-dimensional cosymplectic manifold $M$.

Transvecting (1.9) with $\eta_k \eta^i$ and taking account of (2.2), we obtain

\[(2.4) \quad K_{ji} = \frac{K}{2} \gamma_{ji}.\]

Substituting (2.4) into (1.9), we see that the curvature tensor of $M$ has the form

\[(2.5) \quad K_{kji} = \frac{K}{2} (\gamma_{kk} \gamma_{ji} - \gamma_{kj} \gamma_{ki}),\]

or equivalently

\[(2.6) \quad K_{kji} = \frac{K}{6} (\varphi_{kk} \varphi_{ji} - \varphi_{kj} \varphi_{ji} - 2 \varphi_{kj} \varphi_{ih})\]

by virtue of (1.7).

Thus we have the following

**Theorem 2.1.** The curvature tensor of the 3-dimensional cosymplectic manifold has the form (2.5) or equivalently (2.6).

On the other hand, on previous paper (Eum, [2]), we have defined the cosymplectic Bochner curvature tensor in a 3-dimensional cosymplectic manifold by

\[(2.7) \quad B_{kji} = K_{kji} + \gamma_{k} L_{ji} - \gamma_{j} L_{ki} + L_{kh} \gamma_{ji} - L_{jk} \gamma_{ki} + \varphi_{kk} M_{ji} - \varphi_{kj} M_{ki} + M_{kk} \varphi_{ji} - M_{jk} \varphi_{ki} - 2 (M_{kj} \varphi_{ik} + \varphi_{kj} M_{ih}),\]

where

\[(2.8) \quad L_{ji} = -\frac{1}{6} (K_{ji} - \frac{K}{8} \gamma_{ji}), \quad L = L_{ji} \theta^{ji} = -\frac{K}{8},\]

\[(2.9) \quad M_{ji} = -L_{ji} \varphi_i.\]

Substituting (2.4) into (2.8), we obtain

\[(2.10) \quad L_{ji} = -\frac{K}{16} \gamma_{ji}.\]
and from which

\[(2.11) \quad M_{ji} = -\frac{K}{16} \phi_{ji}.\]

Substituting (2.5), (2.10) and (2.11) into (2.7), we obtain

\[(2.12) \quad B_{kjih} = 0.\]

Thus we have the following

**Theorem 2.2.** In the 3-dimensional cosymplectic manifold, the cosymplectic Bochner curvature tensor vanishes identically.

Taking account of the equation (2.5) and calculating the sectional curvature \(k\) determined by the \(\phi\)-holomorphic section, we easily see that \(k = \frac{K}{2}\).

Taking account of this fact, (1.7), (2.5) and (2.6), we find

\[(2.13) \quad K_{kjih} = \frac{k}{4} (\gamma_{khj} \gamma_ji - \gamma_{jhi} \gamma_{k}\gamma_{ki} + \phi_{kh} \phi_{ji} - \phi_{jih} \phi_{k} - 2 \phi_{k} \phi_{ih}).\]

Thus we have the following

**Theorem 2.3.** In the 3-dimensional cosymplectic manifold, the \(\phi\)-holomorphic sectional curvature is independent of \(\phi\)-holomorphic section at a point and is equal to \(\frac{K}{2}\), \(K\) being the scalar curvature.

### 3. Curvature tensor of 3-dimensional Sasakian manifold

If a normal almost contact metric structure of \(M\) satisfies

\[(3.1) \quad \phi_{ji} = \frac{1}{2} (\partial_j \eta_i - \partial_i \eta_j),\]

then \(M\) is called a Sasakian manifold.

In a \((2n+1)\)-dimensional Sasakian manifold \(M\), we have (Yano, [6])

\[(3.2) \quad V_k \eta_j = \phi_{kj}, \quad V_k \phi_j^h = - \eta^h \phi_{kj} + \eta_j \delta_k^h.\]

The following equations are also satisfied in \(M\) (Yano, [6]):

\[(3.3) \quad K_{kji}^h \eta^f = \delta_k^h \eta_j^f - \delta^h_j \eta_k, \quad \eta^f K_{gij}^h = \eta^g \delta_j^h - \eta_j \delta_i^h,\]

\[(3.4) \quad K_{ji} \eta^f = 2n \eta_j\]

and
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(3.5) \( \eta^i \nabla_i K = 0, \quad \eta^j \nabla_j K = 0. \)

In this section we study on the curvature tensor of the 3-dimensional Sasakian manifold \( M. \)

Transvecting (1.9) with \( \eta_h \delta^h \) and taking account of (3.3) and (3.4), we obtain

(3.6) \[ K_{ji} = b g_{ji} + (2 - b) \eta_j \eta_i, \]

where we have put

(3.7) \[ b = \frac{K}{2} - 1. \]

Substituting (3.6) and (3.7) into (1.9), we see that the curvature tensor of \( M \) has the form

(3.8) \[ K_{kjl} = (1 - b) (g_{ki} \delta_j g_j g_k \delta_l) \]
\[ + (b - 2) (\eta_j \delta_j g_k g_k \delta_l) \eta_i + (g_{ki} \eta_j - g_{kj} \eta_i) \eta_l. \]

Substituting the identity (1.8) into (3.8), we obtain

(3.9) \[ K_{kjih} = (g_{ki} g_j g_k g_{ih}) + 2 \eta_k (g_{ji} \eta_j - g_{jki}) \]
\[ - 2 \eta_j (g_{ki} \eta_k - g_{kji}) - \frac{b}{3} (\varphi_{ki} \varphi_{j} - \varphi_{j} \varphi_{ki} + 2 \varphi_{kj} \varphi_{ih}). \]

Substituting (1.7) into (3.9), we obtain

(3.10) \[ K_{kji} = (1 - b) (1 - \gamma_{ki} \gamma_{ji} - \gamma_{jki}) \]
\[ + \eta_k (g_{ji} \eta_j - g_{jki}) - \eta_j (g_{ki} \eta_j - g_{kji}). \]

Thus we have the following

THEOREM 3.1. The curvature tensor of the 3-dimensional Sasakian manifold has the form (3.8) or (3.9) or equivalently (3.10).

On the other hand, the contact Bochner curvature tensor in the 3-dimensional Sasakian manifold is defined by (Yano, [6])

(3.11) \[ B_{kji}^h = K_{kji}^h + \gamma_k^h L_{ji} - \gamma_j^h L_{ki} + L_k^h \nabla_{ji} - L_j^h \nabla_{ki} \]
\[ + \varphi_k^h M_{ji} - \varphi_j^h M_{ki} + M_k^h \varphi_{ji} - M_j^h \varphi_{ki} \]
\[ - 2 (M_{kj} \varphi_{ki} + M_{kj} M^h_{ij}) + (\varphi_k^h \varphi_{ji} - \varphi_j^h \varphi_{ki} - 2 \varphi_{kj} \varphi_{ih}), \]

where

(3.12) \[ L_{ji} = - \frac{1}{6} \{ K_{ji} + (L + 3) g_{ji} - (L - 1) \eta_j \eta_i \}, \quad L_j^h = L_{ji} g_{ih}, \]
\[ M_{ji} = - L_{ji} \varphi_{i}^h, \quad M_k^h = M_{ki} g_{ih}, \]
and
Substituting (3.6) and (3.7) into (3.12) and (3.13), we obtain

(3.14) \[ L_{ji} = b_1 g_{ji} + b_2 \eta_j \eta_i, \quad M_{ji} = b_3 \phi_{ji}, \]

where we have put

(3.15) \[ b_1 = -\frac{1}{2} \left( \frac{1}{4} b + \frac{1}{2} \right), \quad b_2 = \frac{1}{2} \left( \frac{1}{4} b - \frac{3}{2} \right). \]

Substituting (1.7), (3.10), (3.14) and (3.15) into (3.11), we easily obtain

(3.16) \[ B_{kij}^h = 0. \]

Thus we have the following

**Theorem 3.2.** In the 3-dimensional Sasakian manifold M, the contact Bochner curvature tensor vanishes identically.

Taking account of the definition of \( \phi \)-holomorphic sectional curvature and the equation (3.10), we easily see the following

**Theorem 3.3.** In the 3-dimensional Sasakian manifold, the \( \phi \)-holomorphic sectional curvature is independent of \( \phi \)-holomorphic section at a point and is equal to \( \frac{K}{2} - 2 \), \( K \) being the scalar curvature.

Taking account of above fact, (1.8) and (3.9), we obtain

\[
K_{kijh} = \frac{k+3}{4} (\eta_k g_{ji} - g_{jh} \eta_i) - \frac{k-1}{4} \eta_j (\eta_k g_{ij} - g_{ij} \eta_k) \\
- \eta_j (\eta_k g_{ih} - g_{ih} \eta_k) + \phi_{ki} \phi_{jh} - \phi_{kj} \phi_{ih} + 2 \phi_{kj} \phi_{ih},
\]

\( k \) being the \( \phi \)-holomorphic sectional curvature.

**References**

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