HONAM MATHEMATICAL JOURNAL
Volume 6, Number 1, August. 1984

On the Homology of Algebras*

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§1. INTRODUCTION

The main purpose of this note is to verify that if we take a left Λ-module A for algebra Λ with 1 over a field K and if we define homology module of Λ with coefficients in A (Definition 4.1), two properties concerning homology module hold (Theorem 4.2 and Theorem 4.3). In other words, in §4, we define $H_n(\Lambda, A)$ for a Λ-bimodule A (Definition 4.1) and we show that

(i) for a K-module L and $n>0$ $H_n(\Lambda, \Lambda\otimes_K L\otimes_K \Lambda)=0$

(ii) for Λ-bimodules A, B and C if $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$

is a split exact sequence as K-modules, then

$$\cdots\cdots\rightarrow H_{n+1}(\Lambda, C)\longrightarrow H_n(\Lambda, A)\longrightarrow H_n(\Lambda, B)\longrightarrow H_n(\Lambda, C)\rightarrow\cdots\cdots$$

is a long exact sequence (Theorem 4.2).

On the other hand, we define resolutions $B(\Lambda)$ and $\bar{B}(\Lambda)$. In Theorem 4.3, if Λ is an augmented algebra with augmentation ε: $\Lambda\longrightarrow K$, then for a right Λ-module M and a K-module G,

$$H_n(\Lambda, {}_\varepsilon M)\cong H_n(M\otimes_\Lambda B(\Lambda)),\ H_n(\Lambda, {}_\varepsilon G_\varepsilon)\cong H_n(G\otimes_K \bar{B}(\Lambda)).$$

§2. PRELIMINARIES

Definition 2.1 A *relative abelian category* is a pair of abelian categories $\mathscr{A}$ and $\mathscr{B}$ together with a covariant functor $\square$: $\mathscr{A}\longrightarrow\mathscr{B}$ which is additive, exact and faithful.

Suppose $\square$: $\mathscr{A}\longrightarrow\mathscr{B}$ is a relative abelian category. A short exact sequence

* This reseach was supported by the Korean Ministry of education scholarship foundation for 1984.

$0 \longrightarrow A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \longrightarrow 0$ in $\mathscr{A}$ is said to be *relatively split* (or *□-split*) if the short exact sequence $0 \longrightarrow \Box A \xrightarrow{\Box\kappa} \Box B \xrightarrow{\Box\sigma} \Box C \longrightarrow 0$ is split in $\mathscr{B}$. We denote the class of □-split short exact sequences of $\mathscr{A}$ by $\mathscr{E}$.

A morphism α: $A \longrightarrow B$ of $\mathscr{A}$ with standard factorization $\alpha = \lambda\sigma$ is □-allowable (or $\mathscr{E}$-allowable) if $\Box\lambda$ has a left inverse and $\Box\sigma$ a right inverse in $\mathscr{B}$.

The complex

$$X:\ \cdots\cdots \longrightarrow X_n \xrightarrow{\partial_n} X_{n-1} \longrightarrow \cdots\cdots \longrightarrow X_0 \xrightarrow{\partial_0} X_{-1} \xrightarrow{\partial_{-1}} X_{-2} \longrightarrow \cdots\cdots$$

is an *$\mathscr{A}$-comlpex* if X_n is an object of $\mathscr{A}$ and ∂_n is a morphism of $\mathscr{A}$.

Proposition 2.2 Let □: $\mathscr{A} \longrightarrow \mathscr{B}$ be a relative abelian category.

(i) If X is an $\mathscr{A}$-complex, then $\Box H_n(X) = H_n(\Box X)$ for $n = 0, \pm1, \pm2, \cdots\cdots$

(ii) For an $\mathscr{A}$-complex, $\Box X$ has a contracting homotopy s in $\mathscr{B}$ iff $H_n(X) = 0$ and ∂_n is □-allowable for $n = 0, \pm1, \pm2, \cdots$. When these conditions hold, s can be chosen such that $s^2 = 0$.

Proof. (i): Since □ is exact, (i) holds.

(ii): Suppose that $\Box X$ has a contracting homotopy s such that $\Box\partial_{n+1} s_n + s_{n-1} \Box\partial_n = 1_{\Box X_n}$ for each n. It follows that $H_n(\Box X) = 0$ for all n. Since the covariant functor □ is faithful, $H_n(\Box X) = \Box H_n(X) = 0$ implies that $H_n(X) = 0$. Since $\Box\partial_{n+1} s_n + s_{n-1} \Box\partial_n = 1_{\Box X_n}$, the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \Box\partial_n \longrightarrow \Box X_n \xrightarrow{\Box\partial_n} \mathrm{Im}\ \Box\partial_n \longrightarrow 0$$

is split. This means that ∂_n is □-allowable.

Conversely, assume that the sequence X: $\cdots \longrightarrow X_{n+1} \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots$ is exact and all ∂_n are □-allowable. Since □ is exact

$$\Box X: \cdots \longrightarrow \Box X_{n+1} \xrightarrow{\Box\partial_{n+1}} \Box X_n \xrightarrow{\Box\partial_n} \Box X_{n-1} \longrightarrow \cdots$$

is exact. On the other hand, since all ∂_n is allowable,

$$\Box X_n \cong \mathrm{Ker}\ \Box\partial_n \oplus \mathrm{Im}\ \Box\partial_n = \mathrm{Im}\ \Box\partial_{n+1} \oplus \mathrm{Im}\ \Box\partial_n.$$

Hence there is the inclusion map s'_{n-1}: $\mathrm{Im}\ \Box\partial_n \longrightarrow \Box X_n$. Since $\Box X_{n-1} \cong \mathrm{Im}\ \Box\partial_n \oplus \mathrm{Im}\ \Box\partial_{n-1}$, we get the extension s_{n-1} of s'_{n-1}. Then, it is easy to know that

$$s_n s_{n-1}=0,\ \square\partial_{n+1}s_n+s_{n-1}\square\partial_n=1_{\square X_n}.$$

Therefore $s=\{s_n\}$ is a contracting homotopy of $\square X$ such that $s^2=0$. ///

For each morphism α: $\mathscr{A}\longrightarrow\mathscr{B}$ in $\mathscr{A}$ such that the exact sequence

$$0\longrightarrow \mathrm{Ker}\,\square\alpha\longrightarrow\square A\xrightarrow{\square\alpha}\square B\longrightarrow 0$$

is split in $\mathscr{B}$, and for each morphism β: $P\longrightarrow B$ in $\mathscr{A}$ if there is a morphism γ: $P\longrightarrow A$ in $\mathscr{A}$ such that $\beta=\alpha\gamma$, then P is called a $\square$-*allwable projective object* in $\mathscr{A}$. A $\square$-allowable projective object will also be called a *relative projective object* for $\square$.

Definition 2.3 *A resolvent pair*($\mathscr{A}$, $\mathscr{B}$) of categories is a relative abelian category $\square$: $\mathscr{A}\longrightarrow\mathscr{B}$ together with

(i) a covariant functor F: $\mathscr{B}\longrightarrow\mathscr{A}$,

(ii) a natural transformation e: $1_{\mathscr{B}}\longrightarrow\square F$ such that each morphism u: $M\longrightarrow\square A$ in $\mathscr{B}$ has the factorization $u=\square\alpha e_M$ for a unique morphism α: $F(M)\longrightarrow A$ of $\mathscr{A}$, where $1_{\mathscr{B}}$ is the identity functor of $\mathscr{B}$.

For a resolvent pair ($\mathscr{A}$, $\mathscr{B}$) of categories, take an object A of $\mathscr{A}$ and an object M of $\mathscr{B}$, then we have the canonical isomorphism

$$e^*:\ \mathrm{Hom}_{\mathscr{A}}(FM, A)\cong\mathrm{Hom}_{\mathscr{B}}(M, \square A)$$

of abelian groups which is defined by $e^*(\alpha)=\square\alpha e_M$. We can also prove that each $F(M)$ is a relative projective object for $\square$. *A relative free complex* ε_X: $X\longrightarrow A$ over A in $\mathscr{A}$ is an $\mathscr{A}$-complex

$$X\longrightarrow A:\ \cdots\cdots\longrightarrow X_n\xrightarrow{\partial_n}X_{n-1}\longrightarrow\cdots\cdots\longrightarrow X_1\xrightarrow{\partial_1}X_0\xrightarrow{\varepsilon_X}A$$

in $\mathscr{A}$ such that for each n ($n=0, 1, 2, \cdots$) X_n is of the form $F(M_n)$. An *allowable resolution* ε_Y: $Y\longrightarrow B$ in $\mathscr{A}$ is an $\mathscr{A}$-complex over B such that $\square\varepsilon_Y$: $\square Y\longrightarrow\square B$ has a contracting homotopy s with $s^2=0$. Then we can easily prove the following.

Proposition 2.4 If a relative abelian category $\square$: $\mathscr{A}\longrightarrow\mathscr{B}$ is a resolvent pair of categories for two functors F and F', then there is a unique natural transformation η: $F\longrightarrow F'$ with $\square\eta e=e'$, where e: $1_{\mathscr{B}}\longrightarrow\square F$ and e': $1_{\mathscr{B}}\longrightarrow\square F'$.

§3. BAR RESOLUTIONS

Let a relative abelian category $\square$: $\mathscr{A}\longrightarrow\mathscr{B}$ be a resolvent pair of categories. We put $F\square=F_\square$, $(F\square)\cdots(F\square)$ (n-times)$=F_\square^n$, and define

$$\beta_n(C)=\beta_n(\mathscr{A},\mathscr{B};C)=F_\square^n F_\square C \text{ for each object } C \text{ of } \mathscr{A} \text{ and } n=0,1,2,\cdots.$$

Then there is a sequence

$$\square C\xrightarrow{s_{-1}}\square\beta_0(C)\xrightarrow{s_0}\square\beta_1(C)\longrightarrow\cdots\cdots\longrightarrow\square\beta_n(C)\xrightarrow{s_n}\square\beta_{n+1}(C)\longrightarrow\cdots$$

in $\mathscr{B}$, where $s_{-1}=e_{\square C}$, $s_n=e_{\square\beta_n(C)}(n\geq 0)$.

Proposition 3.1 There are unique morphisms

$$\varepsilon:\ \beta_0(C)\longrightarrow C,\ \ \partial_n:\ \beta_n(C)\longrightarrow\beta_{n-1}(C)\ \ (n\geq 1)$$

of $\mathscr{A}$ such that $\beta(C)=\{\beta_n(C)\,|\,n=0,1,2,\cdots\}$ is a relatively free allowable resolution of C with s as its contracting homotopy in $\mathscr{B}$.

Proof. By the isomorphism $\mathrm{Hom}_{\mathscr{A}}(F(\square C),C)\cong\mathrm{Hom}_{\mathscr{B}}(\square C,\square C)$, ε: $F\square C=\beta_0(C)\longrightarrow C$ corresponds to morphism $1_{\square C}$: $\square C\longrightarrow\square C$. Hence $\square\varepsilon s_{-1}=1_{\square C}$, and ε is $\square$-allowable. ∂_1: $\beta_1(C)\longrightarrow\beta_0(C)$ is uniquely determined by the canonical isomorphism;

$$\begin{array}{ccc}\mathrm{Hom}_{\mathscr{A}}(\beta_1(C),\ \beta_0(C)) & \cong & \mathrm{Hom}_{\mathscr{B}}(\square\beta_0(C),\ \square\beta_0(C)) \\ \Cup & & \Cup \\ \partial_1 & \rightsquigarrow & 1-s_{-1}\square\varepsilon\end{array}$$

Then, $\square\partial_1 s_0=\square\partial_1 e_{\square\beta_0(C)}=1-s_{-1}\square\varepsilon$, and $\square\varepsilon\square\partial_1=\square(\varepsilon\partial_1)=0$. Since $\square$ is faithful, $\varepsilon\partial_1=0$. By induction, $\partial_n:\beta_n(C)\longrightarrow\beta_{n-1}(C)$ $(n\geq 1)$ is uniquely determined by the isomorphism as follows;

$$\begin{array}{ccc}\mathrm{Hom}_{\mathscr{A}}(\beta_n(C),\ \beta_{n-1}(C)) & \cong & \mathrm{Hom}_{\mathscr{B}}(\square\beta_{n-1}(C),\ \square\beta_{n-1}(C)) \\ \Cup & & \Cup \\ \partial_n & \rightsquigarrow & 1-s_{n-2}\square\partial_{n-1}\end{array}$$

Then, $\square\partial_n s_{n-1}=1-s_{n-1}\square\partial_{n-1}$. Hence, by the same way as above we can prove that $\partial_n\partial_{n+1}=0(n\geq 1)$. Furthermore, since $\square\partial_{n+1}s_n\square\partial_{n+1}=\square\partial_{n+1}$, $\partial_{n+1}(n\geq 0)$ is $\square$-allowable. ///
(Note that $\beta(\mathscr{A},\mathscr{B};C)=\{\beta_n(C)\}$ is called the *un-normalized bar resolution* of C.)

Next, for a resolvent pair $(\mathscr{A},\mathscr{B})$ and for each object M of $\mathscr{B}$, consider the exact sequence

$$M \xrightarrow{e_M} \Box FM \xrightarrow{p_M} \bar{F}M = \Box FM/e_M(M) \longrightarrow 0$$

in $\mathscr{B}$. Then it follows that

(i) $\bar{F}$: $\beta \longrightarrow \beta$ is a covariant functor,

(ii) p: $\Box F \longrightarrow \bar{F}$ is a natural transformation,

(iii) if $s_M = e_{\bar{F}M} \cdot p_M$, then s_M is a natural transformation $\Box F \longrightarrow \Box F\ \bar{F}$ ([11]).

Then, for each object C of $\mathscr{A}$, from the exact sequence

$$\bar{F}\Box C \longrightarrow F\bar{F}\Box C \longrightarrow \bar{F}\bar{F}\Box C = \bar{F}^2 \Box\, C \longrightarrow 0$$

in $\mathscr{B}$, we get an object $\bar{F}^2 \Box C$ of $\mathscr{B}$. Put

$$B_n(C) = B_n(\mathscr{A}, \mathscr{B}; C) = F\bar{F}^n \Box C \text{ and } M_n = \bar{F}^n \Box C \quad (n = 0, 1, 2, \cdots).$$

Consider the sequene

$$\Box C \xrightarrow{s_{-1}} \Box B_0(C) \xrightarrow{s_0} \Box B_1(C) \xrightarrow{s_1} \cdots\cdots$$

where $s_{-1} = e_{\Box C}$: $\Box C \longrightarrow \Box F \Box C$ and $s_n = e_{\bar{F}M_n} \cdot p_{M_n}$: $\Box FM_n = \Box B_n(C) \longrightarrow \Box F\bar{F}M_n = \Box B_{n+1}(C)$ $(n \geq 0)$. By this construction we easily see that $s_{n+1}s_n = 0$ $(n \geq -1)$.

Proposition 3.2 There are unique morphisms

$$\varepsilon:\ B_0(C) \longrightarrow C,\quad \partial_n:\ B_n(C) \longrightarrow B_{n-1}(C) \quad (n \geq 1)$$

such that $B(\mathscr{A}, \mathscr{B}; C) = \{B_n(C)\}$ is an $\mathscr{A}$-complex and a relatively free allowable resolution of C with s as contracting homotopy in $\mathscr{B}$ such that $s^2 = 0$.

Proof. By the isomorphism $\mathrm{Hom}_{\mathscr{A}}(B_0(C), C) \cong \mathrm{Hom}_{\mathscr{B}}(\Box C, \Box C)$, ε: $B_0(C) \longrightarrow C$ corresponds to the morphism $1_{\Box C}$: $\Box C \longrightarrow \Box C$. Hence $\Box \varepsilon s_{-1} = 1_{\Box C}$. By induction, we construct ∂_n with

$$\Box \varepsilon s_{-1} = 1_{\Box C},\ \Box \partial_1 s_0 = 1 - s_{-1}\ \Box \varepsilon,\ \Box \partial_n s_{n-1} = 1 - s_{n-2}\ \Box \partial_{n-1} \quad (n > 1)$$

for a contracting homotopy s. Given $\partial_1, \cdots\cdots, \partial_{n-1}$ satisfying the above equality,

$$(1 - s_{n-2}\ \Box \partial_{n-1}) s_{n-2} = s_{n-2} - s_{n-2}(1 - s_{n-3}\ \Box \partial_{n-2}) = s_{n-2}s_{n-3}\ \Box \partial_{n-1} = 0;$$

since $s_{n-2}=e_{\bar{F}^{n-1}(\Box C)}\cdot p_{\bar{F}^{n-2}(\Box C)}$ with $p_{\bar{F}^{n-2}(\Box C)}$ epic, $(1-s_{n-2}\Box\partial_{n-1})\,e_{\bar{F}^{n-1}(\Box C)}=0$.

Therefore $1-s_{n-2}\Box\partial_{n-1}$ factors through Cok $e_{\bar{F}^{n-2}(\Box C)}=p_{\bar{F}^{n-2}(\Box C)}$ as $1-s_{n-2}\Box\partial_{n-1}=\gamma_n\cdot p_{\bar{F}^{n-1}(\Box C)}$ and γ_n is unique. By the isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(F\bar{F}^n(\Box C),\ F\bar{F}^{n-1}(\Box C))\cong\mathrm{Hom}_{\mathscr{B}}(\bar{F}^n(\Box C),\ \Box F\bar{F}^{n-1}(\Box C))$$

∂_n corresponds to the morphism γ_n: $\bar{F}^n(\Box C)\longrightarrow\Box F\bar{F}^{n-1}(\Box C)$.
Then, since $s_{n-1}=e_{\bar{F}^n(\Box C)}\cdot p_{\bar{F}^{n-1}(\Box C)}$ we see that $\Box\partial_n s_{n-1}=1-s_{n-2}\Box\partial_{n-1}$.

It is clear that ε is $\Box$-allowable. Moreover, since $p_{\Box C}$ is epic and $\Box\varepsilon(1-s_{-1}\Box\varepsilon)=0$, we have $\Box\varepsilon\gamma_1$: $\bar{F}(\Box C)\longrightarrow\Box C$ which is a zero morphism. Therefore $\varepsilon\partial_1$ corresponds to the zero morphism $\Box\varepsilon\gamma_1$ in the isomorphism $\mathrm{Hom}_{\mathscr{A}}(F\bar{F}(\Box C),C)\cong\mathrm{Hom}_{\mathscr{B}}(\bar{F}(\Box C),\ \Box C)$. It follows that $\varepsilon\partial_1=0$. By the same way as above we can prove that $\partial_{n-1}\partial_n=0$ $(n\geq 2)$. Since

$$\Box\partial_n s_{n-1}\Box\partial_n=(1-s_{n-2}\Box\partial_{n-1})\Box\partial_n=\Box\partial_n\ (n\geq 1),$$

∂_n is $\Box$-allowable. ///
Note that $\mathscr{B}(\mathscr{A},\mathscr{B};C)$ is called *normalized bar resolution* of C.

Example 3.3 Let K be a field and let Λ be an algebra with 1 over K. The class ${}_\Lambda\mathscr{M}$ of all left Λ-modules is a category whose morphisms are all left Λ-module homomorphisms, and the class ${}_K\mathscr{M}_K$ of all K-modules is a category whose morphisms are all K-module homomorphisms. Define

$$\Box:\ {}_\Lambda\mathscr{M}\longrightarrow{}_K\mathscr{M}_K$$

as the usual neglect functor. For each a K-module M and $m\in M$ define

$$F(M)=\Lambda\otimes_K M,\ e_M:\ M\longrightarrow\Lambda\otimes_K M\,(m\longmapsto 1\otimes m).$$

Then $({}_\Lambda\mathscr{M},\ {}_K\mathscr{M}_K)$ is a resolvent pair of categories and $F(M)$ is relatively projective in ${}_\Lambda\mathscr{M}$.

Proof. At first, we shall prove that for each object A of ${}_\Lambda\mathscr{M}$ and object M of ${}_K\mathscr{M}_K$,

$$e_M^*:\ \mathrm{Hom}_{{}_\Lambda\mathscr{M}}(F(M),\ A)\cong\mathrm{Hom}_{{}_K\mathscr{M}_K}(M,\Box A),$$

where for each $\alpha\in\mathrm{Hom}_{{}_\Lambda\mathscr{M}}(F(M),A)$, $e_M^*(\alpha)=\Box\alpha e_M$. Since α is a left Λ-module homomorphism, for each $\lambda\otimes m\in\Lambda\otimes_K M$, $\alpha(\lambda\otimes m)=\alpha(\lambda(1\otimes m))=\lambda\alpha(1\otimes m)$, and thus for two left Λ-module homomorphisms α_1, α_2: $\Lambda\otimes_K M\longrightarrow A$ if $\alpha_1\neq\alpha_2$, then there is at least one element $m\in M$ such that $\alpha_1(1\otimes m)\neq\alpha_2(1\otimes m)$. Therefore e_M^* is injective. Take a morphism of ${}_K\mathscr{M}_K$ f: $M\longrightarrow\square A$ and define $\tilde{f}$: $\Lambda\otimes_K M\longrightarrow A$ by $\tilde{f}(\lambda\otimes m)=\lambda f(m)$ for each $\lambda\otimes m\in\Lambda\otimes_K M$. Then for each $\lambda_1\in\Lambda$,

$$\tilde{f}(\lambda_1(\lambda\otimes m))=\tilde{f}(\lambda_1\lambda\otimes m)=\lambda_1\lambda f(\mathrm{m})=\lambda_1\tilde{f}(\lambda\otimes m)$$

and thus $\tilde{f}$ is a morphism of ${}_\Lambda\mathscr{M}$. In this case, it is clear that $\square\tilde{f}e_M=f$. Therefore e_M^* is surjective.

Next, we shall prove that $F(M)$ is relatively projective of ${}_\Lambda\mathscr{M}$. Let σ: $B\longrightarrow C$ be a morphism of ${}_\Lambda\mathscr{M}$ such that the sequence

$$0\longrightarrow\mathrm{Ker}\,\square\sigma\longrightarrow\square B\xrightarrow{\square\sigma}\square C\longrightarrow 0$$

is split in ${}_K\mathscr{M}_K$. Then there is a morphism κ of ${}_K\mathscr{M}_K$ such that $\square\sigma\cdot\kappa=1_{\square C}$. For each morphism τ: $F(M)\longrightarrow C$ in ${}_\Lambda\mathscr{M}$, $\kappa\square\tau\cdot e_M$: $M\longrightarrow\square B$. In the isomorphisms $\mathrm{Hom}_{{}_\Lambda\mathscr{M}}(F(M),B)\cong\mathrm{Hom}_{{}_K\mathscr{M}_K}(M,\square B)$, let us take μ: $F(M)\longrightarrow B$ corresponds to the morphism $\kappa\square\tau\cdot e_M$. Then we have $\square(\sigma\mu)\cdot e_M=\square\tau\cdot e_M$ and thus $\sigma\mu=\tau$. ///

Since $F_\square(C)=F(\square C)=\Lambda\otimes_K\square C=\Lambda\otimes_K C$ for each object C of ${}_\Lambda\mathscr{M}$, the un-normalized bar resolution $\beta({}_\Lambda\mathscr{M},\ {}_K\mathscr{M}_K;C)=\beta(\Lambda,C)$ of C has

$$\beta_n(C)=\beta_n(\Lambda,C)=\beta_n({}_\Lambda\mathscr{M},\ {}_K\mathscr{M}_K;C)=\Lambda\otimes_K\Lambda^n\otimes_K C,$$

where $\Lambda^n=\Lambda\otimes_K\Lambda\otimes_K\cdots\otimes_K\Lambda$ (n-times). For any element $\lambda\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\otimes c\in\beta_n(\Lambda,C)$ $=\beta_n(C)$, the contracting homotopy $s=\{s_n\}$ is defined as in Proposition 3.1

$$s_{-1}(c)=e_{\square C}(c)=1\otimes c$$
$$s_n(\lambda\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\otimes c)=1\otimes(\lambda\otimes\lambda_1\otimes\cdots\otimes\lambda_n)\otimes c\quad(n\geq 0)\quad\cdots\cdots\cdots\cdots(\mathrm{I})$$

We also define left Λ-module homomorphisms ε and ∂_n ($n\geq 1$) by

$$\varepsilon(\lambda\otimes c)=\lambda c$$
$$\partial_n(\lambda\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\otimes c)=\lambda\lambda_1\otimes(\lambda_2\otimes\cdots\otimes\lambda_n)\otimes c$$
$$+\sum_{i=1}^{n-1}(-1)^i\lambda\otimes(\lambda_1\otimes\cdots\otimes\hat{\lambda}_i\otimes\lambda_i\lambda_{i+1}\otimes\cdots\otimes\lambda_n)\otimes c$$
$$+(-1)^n\lambda\otimes(\lambda_1\otimes\cdots\otimes\lambda_{n-1})\otimes\lambda_n c\quad\cdots\cdots\cdots\cdots(\mathrm{II})$$

where $\hat{\lambda}_i$ means that λ_i is omitted. Then ε and ∂_n are uniquely determined and they satisfy the equations:

$$\Box\varepsilon s_{-1}=1_{\Box C}, \qquad \Box\partial_1 s_0=1-s_{n-1}\Box\varepsilon,$$
$$\Box\partial_{n+1}\, s_n=1-s_{n-1}\,\Box\partial_n(n\geq 1),\ \ \varepsilon\partial_1=0=\partial_n\partial_{n+1}(n\geq 1), \quad \cdots\cdots\cdots\cdots(\text{III})$$

Hence, by Proposition 3.1, $\beta(\Lambda,C)$ is a relatively free allowable resolution of C with s as its contracting homotopy in ${}_K\mathscr{M}_K$ such that $s^2=0$. Next, we want to construct the normalized bar resolution of C. Since Λ is an algebra with 1 over K, there is the K-module homomrphism I: $K\longrightarrow\Lambda$ such that $I(1)=1$. Hence we have the exact sequence of K-modules:

$$0\longrightarrow K\xrightarrow{I}\Lambda\longrightarrow\Lambda/I(K)=\Lambda/K\longrightarrow 0$$

Since C is a K-module by I: $K\longrightarrow\Lambda$, we have the exact sequence of K-modules:

$$C=K\otimes_K C\xrightarrow{I\otimes 1_C}\Lambda\otimes_K C\longrightarrow(\Lambda/K)\otimes_K C\longrightarrow 0$$

and thus $\overline{F}(\Box C)\cong(\Lambda/K)\otimes_K\Box C=(\Lambda/K)\otimes_K C$. Therefore

$$B_n(C)=B_n(\Lambda,C)=B_n({}_\Lambda\mathscr{M},\ {}_K\mathscr{M}_K;C)=F\overline{F}^n\Box C=\Lambda\otimes_K(\Lambda/K)^n\otimes_K C.$$

We shall denote elements of $B_n(\Lambda,C)$ by

$$\lambda[\lambda_1|\cdots|\lambda_n]c=\lambda\otimes(\lambda_1+K)\otimes\cdots\otimes(\lambda_n+K)\otimes c$$

In particular, elements of $B_0(\Lambda,C)$ are written as $\lambda\ [\ ]\ c$.
As in (I) and (II), s_n: $\Box B_n(\Lambda,C)\longrightarrow\Box B_{n+1}(\Lambda,C)$ and ∂_n: $B_n(\Lambda,C)\longrightarrow B_{n-1}(\Lambda,C)$ are defined by setting

$$s_{-1}(c)=1\ [\ ]\ c,\ \ s_n(\lambda[\lambda_1|\cdots|\lambda_n]c)=1[\lambda|\lambda_1|\cdots|\lambda_n]c,\ \ \varepsilon(\lambda[\ ]c)=\lambda c,$$
$$\partial_n(\lambda[\lambda_1|\cdots|\lambda_n]c)=\lambda\lambda_1[\lambda_2|\cdots|\lambda_n]c+\sum_{i=1}^{n-1}(-1)^i\lambda[\lambda_1|\cdots|\hat{\lambda}_i|\lambda_i\lambda_{i+1}|\cdots|\lambda_n]c$$
$$+(-1)^n\lambda[\lambda_1|\cdots|\lambda_{n-1}]\lambda_n c.$$

Then, these morphisms satisfy (III). Therefore, by Proposition 3.2, $B(\Lambda,C)=\{B_n(\Lambda,C)\}$ is a relatively free allowable resolution of C with contracting homotopy s in ${}_K\mathscr{M}_K$ such that $s^2=0$. Note that $\lambda[1|\lambda_2|\cdots|\lambda_n]c=0$ in $B_n(\Lambda,C)$.

Theorem 3.4 Two complexes $B(\Lambda,C)$ and $\beta(\Lambda,C)$ over a left Λ-module C are chain equivalent.

Proof. Define η_n: $\beta_n(C) \longrightarrow B_n(C)$ by $\eta_n(\lambda \otimes (\lambda_1 \otimes \cdots \otimes \lambda_n) \otimes c) = \lambda[\lambda_1|\cdots|\lambda_n]c$. Then it is obvious that η_n is a Λ-module chain transformation lifting 1_C: $C \longrightarrow C$. Define $\mathscr{E}_n$: $B_n(C) \longrightarrow \beta_n(C)$ as follows:

(i) $n=0$; $\mathscr{E}_0$: $\Lambda \otimes_K C \longrightarrow \Lambda \otimes_K C$ by $\lambda[\ ]c \longmapsto \lambda \otimes c$.

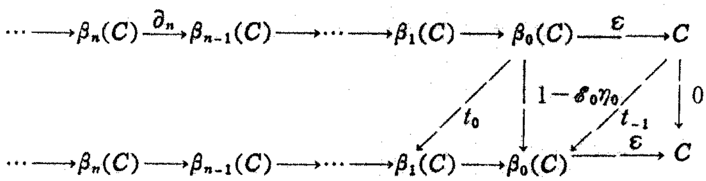
(ii) By induction, $\mathscr{E}_n$ is defined by $\square \mathscr{E}_n e_n' = s_{n-1} \square \mathscr{E}_{n-1} \square \partial_n' e_n'$,

where $e_n' = e_{\bar{F}(\square C)}$ and ∂_n' is the boundary of $B(\Lambda, C)$. That is, $\mathscr{E}_n$ corresponds to the K-module homomorphism $s_{n-1} \square \mathscr{E}_{n-1} \square \partial_n' e_n'$: $\bar{F}^n \square C \longrightarrow \square \beta_n(C)$ by the isomorphism $\operatorname{Hom}_{\Lambda}\mathscr{M}(B_n(C), \beta_n(C)) \cong \operatorname{Hom}_K \mathscr{M}_K(\bar{F}^n \square C, \square \beta_n(C))$. Then, it is easy to prove that $\{\mathscr{E}_n\}$ is a Λ-module chain transformation lifting $1_C: C \longrightarrow C$. We want to prove that $\eta\mathscr{E} \simeq 1_{B(C)}$ and $\mathscr{E}\eta \simeq 1_{\beta(C)}$. For each $[\lambda_1|\cdots|\lambda_n]c \in \bar{F}^n \square C$, since

$$
\begin{aligned}
\square \eta_n s_{n-1} \square \mathscr{E}_{n-1} \square \partial_n' e_n'([\lambda_1|\cdots|\lambda_n]c) &= \square \eta_n s_{n-1} \square \mathscr{E}_{n-1}(\lambda_1[\lambda_2|\cdots|\lambda_n]c \\
&\quad + \sum_{i=1}^{n-1}(-1)^i 1[\lambda_1|\cdots|\hat{\lambda}_i|\lambda_i\lambda_{i+1}|\cdots|\lambda_n]c + (-1)^n 1[\lambda_1|\cdots|\lambda_{n-1}]\lambda_n c) \\
&= \square \eta_n(1 \otimes (\lambda_1 \otimes \cdots \otimes \lambda_n) \otimes c + \sum_{i=1}^{n-1}(-1)^i\ 1 \otimes (1 \otimes \lambda_1 \otimes \cdots \otimes \hat{\lambda}_i \otimes \lambda_i \lambda_{i+1} \\
&\quad \otimes \cdots \otimes \lambda_n) \otimes c + (-1)^n 1 \otimes (1 \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n-1}) \otimes \lambda_n c) \\
&= 1[\lambda_1|\cdots|\lambda_n]c = e_n'([\lambda_1|\cdots|\lambda_n]c),
\end{aligned}
$$

we have $\eta_n \mathscr{E}_n = 1_{B_n(C)}$ $(n \geq 0)$.

Consider the diagram



where $t_{-1}=0$. Since $\varepsilon(1-\mathscr{E}_0\eta_0 - t_{-1}\varepsilon)$ $(\lambda \otimes c) = \lambda c - \lambda c = 0$, $\operatorname{Im}(1-\mathscr{E}_0\eta_0 - t_{-1}\varepsilon) \subset \operatorname{Ker} \varepsilon$. As in example 3.3, since $\beta_n(C)$ $(n \geq 0)$ is relatively projective and

$$0 \longrightarrow \operatorname{Ker} \square\partial_1 \longrightarrow \square\beta_1(C) \longrightarrow \operatorname{Ker} \square\varepsilon \longrightarrow 0$$

is split and exact as K-modules, there is a left Λ-module homomorphism t_0: $\beta_0(C) \longrightarrow \beta_1(C)$ such that

$$\partial_1 t_0 = 1 - \mathscr{E}_0\eta_0 - t_{-1}\varepsilon.$$

By induction we can define a left Λ-module homomorphism $t_n: \beta_n(C) \longrightarrow \beta_{n+1}(C)$ such that

$$\partial_{n+1}\, t_n + t_{n-1}\partial_n = 1 - \mathscr{E}_n\eta_n \ (n \geq 1).$$

This implies that $t: 1 \simeq \mathscr{E}\eta$. ///

§4. HOMOLOGY OF ALGEBRAS

Let K be a field with characteristic$\neq 2$, and let Λ be an algebra with 1 over K. Two Λ-bimodules A and B have a bimodule tensor product $A \otimes_{\Lambda\text{-}\Lambda} B$ which is obtained from the tensor product $A \otimes_K B$ by the equations

$$a\lambda \otimes b = a \otimes \lambda b,\ \ \lambda a \otimes b = a \otimes b\lambda \quad \cdots\cdots (\text{IV})$$

for $a \in A$, $b \in B$ and $\lambda \in \Lambda$. For a Λ-bimodule A and a K-module M we have isomorphisms

$$\alpha:\ \Lambda \otimes_\Lambda A \cong A, \qquad \theta:\ A \otimes_{\Lambda\text{-}\Lambda} (\Lambda \otimes_K M \otimes_K \Lambda) \cong A \otimes_K M \quad \cdots\cdots (\text{V})$$

by $\alpha(\lambda \otimes a) = \lambda a$ and $\theta(a \otimes (\lambda_1 \otimes m \otimes \lambda_2)) = \lambda_2 a \lambda_1 \otimes m$. The inverses are given by $\alpha^{-1}(a) = 1 \otimes a$ and $\theta^{-1}(a \otimes m) = a \otimes (1 \otimes m \otimes 1)$.

Definition 4.1 Let A be a Λ-bimodule. The (*Hochschild*) *homology modules* of a K-algebra Λ with coefficients in A are defined via the bar (un-)normalized resolution $B(\Lambda, \Lambda)$ (or $\beta(\Lambda, \Lambda)$) to be K-modules

$$H_n(\Lambda, A) = H_n(A \otimes_{\Lambda\text{-}\Lambda} B(\Lambda, \Lambda))\ (\cong H_n(A \otimes_{\Lambda\text{-}\Lambda} \beta(\Lambda, \Lambda)))\ (n = 0, 1, 2, \cdots).$$

In particular,

$$A \otimes_{\Lambda\text{-}\Lambda} \beta_n(\Lambda, \Lambda) = A \otimes_{\Lambda\text{-}\Lambda} (\Lambda \otimes_K \Lambda^n \otimes_K \Lambda) \cong A \otimes_K \Lambda^n$$

by (V), and in this case

$$\partial_n:\ A \otimes_{\Lambda\text{-}\Lambda} \beta_n(\Lambda, \Lambda) \longrightarrow A \otimes_{\Lambda\text{-}\Lambda} \beta_{n-1}(\Lambda, \Lambda)$$

is defined by $\partial_n: A \otimes_K \Lambda^n \longrightarrow A \otimes_K \Lambda^{n-1}$ such that $\partial_n = d_0 - d_1 + \cdots + (-1)^n d_n$, where d_i: $A \otimes_K \Lambda^n \longrightarrow A \otimes_K \Lambda^{n-1}$ is defined by

$$d_i(a \otimes \lambda_1 \otimes \cdots \otimes \lambda_n) = \begin{cases} a\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n & \text{if } i = 0 \\ a \otimes \lambda_1 \otimes \cdots \otimes \hat{\lambda}_i \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_n & \text{if } 1 \leq i < n \\ \lambda_n a \otimes \lambda_1 \otimes \cdots \otimes \lambda_{n-1} & \text{if } i = n. \end{cases}$$

Since $\partial_1(a\otimes\lambda)=a\lambda-\lambda a$ for each $a\otimes\lambda\in A\otimes_K\Lambda$, we have the isomorphism

$$H_0(\Lambda,A)\cong A/\{\lambda a-a\lambda\mid a\in A,\ \lambda\in\Lambda\},$$

which is the quotient of A by the K-submodule generated by all elements $\lambda a-a\lambda$.

Theorem 4 2 Under the above situation we have the following:

(i) For a K-module L and $n>0$ $\quad H_n(\Lambda,\Lambda\otimes_K L\otimes_K\Lambda)=0$,

(ii) For Λ-bimodules A, B and C if $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ is a split exact sequence as K-modules, then there is for each $n>0$ a connecting homomorphism E_n: $H_n(\Lambda,C)\longrightarrow H_{n-1}(\Lambda,A)$ such that the long sequence

$$\cdots\longrightarrow H_{n+1}(\Lambda,C)\xrightarrow{E_{n+1}}H_n(\Lambda,A)\longrightarrow H_n(\Lambda,B)\longrightarrow H_n(\Lambda,C)\longrightarrow\cdots$$

is exact.

Proof. (i): Note that ∂_n: $(\Lambda\otimes_K L\otimes_K\Lambda)\otimes_K\Lambda^n\longrightarrow(\Lambda\otimes_K L\otimes_K\Lambda)\otimes\Lambda^{n-1}$ is defined by

$$\begin{aligned}\partial_n((\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n))&=(\lambda\otimes l\otimes\lambda'\lambda_1)\otimes(\lambda_2\otimes\cdots\otimes\lambda_n)\\&+\sum_{i=1}^{n-1}(-1)^i(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\hat{\lambda}_i\otimes\lambda_i\lambda_{i+1}\otimes\cdots\otimes\lambda_n)\\&+(-1)^n(\lambda_n\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_{n-1}).\end{aligned}$$

As is well-known

$$\nu:\ (\Lambda\otimes_K L\otimes_K\Lambda)\otimes_K\Lambda^n\cong(\Lambda\otimes_K L\otimes_K\Lambda)\otimes_K\Lambda^n\cdots\cdots\cdots\cdots(\text{VI})$$

by $\nu((\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n))=(\lambda\otimes l\otimes\lambda_1)\otimes(\lambda_2\otimes\cdots\otimes\lambda_n\otimes\lambda')$.

Assume that

$$(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\in\ker\partial_n\subset(\Lambda\otimes_K L\otimes_K\Lambda)\otimes_K\Lambda^n.$$

Take $(\lambda\otimes l\otimes 1)\otimes(\lambda'\otimes\lambda_1\otimes\cdots\otimes\lambda_n)\in(\Lambda\otimes_K L\otimes_K\Lambda)\otimes_K\Lambda^{n+1}$, then

$$\begin{aligned}\partial_{n+1}((\lambda\otimes l\otimes 1)\otimes(\lambda'\otimes\lambda_1\otimes\cdots\otimes\lambda_n))&=(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n)\\&-\{(\lambda\otimes l\otimes 1)\otimes(\lambda'\lambda_1\otimes\lambda_2\otimes\cdots\otimes\lambda_n)\\&+\sum_{i=1}^{n-1}(-1)^i(\lambda\otimes l\otimes 1)\otimes(\lambda'\otimes\lambda_1\otimes\cdots\otimes\hat{\lambda}_i\otimes\lambda_i\lambda_{i+1}\otimes\cdots\otimes\lambda_n)\\&+(-1)^n(\lambda_n\lambda\otimes l\otimes 1)\otimes(\lambda'\otimes\lambda_1\otimes\cdots\otimes\lambda_{n-1})\}.\end{aligned}$$

But, by the isomorphism ν, the terms in the above { } corresponds to

$$\{(\lambda\otimes l\otimes\lambda'\lambda_1)\otimes(\lambda_2\otimes\cdots\otimes\lambda_n)+\sum_{i=1}^{n-1}(-1)^i(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\hat{\lambda}_i\otimes\lambda_i\lambda_{i+1}\otimes\cdots\otimes\lambda_n)$$
$$+(-1)^n(\lambda_n\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_{n-1})\}\otimes 1$$
$$=\partial_n(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n))\otimes 1.$$

Since $\partial_n((\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n))=0,$

$$\partial_{n+1}((\lambda\otimes l\otimes 1)\otimes(\lambda'\otimes\lambda_1\otimes\cdots\otimes\lambda_n))=(\lambda\otimes l\otimes\lambda')\otimes(\lambda_1\otimes\cdots\otimes\lambda_n).$$

This implies that Ker $\partial_n=\mathrm{Im}\ \partial_{n+1}$ $(n\geq 1)$. Hence the proof is complete.

(ii) By hypothesis, the sequence

$$0\longrightarrow A\otimes_K\Lambda^n\longrightarrow B\otimes_K\Lambda^n\longrightarrow C\otimes_K\Lambda^n\longrightarrow 0$$

is exact as K-modules for each $n\geq 0$. Hence there is the desired connecting homomorphism. ///

Let Λ be an algebra with 1 over a field K. If ε: $\Lambda\longrightarrow K$ is an augmented algebra, then a right Λ-module M is a Λ-bimodule, written ${}_\varepsilon M$, by the way such that for $\lambda_1,\lambda_2\in\Lambda$ and $m\in M$ $\lambda_1 m\lambda_2=\varepsilon(\lambda_1)m\lambda_2$.

Furthermore, a K-module G is a Λ-bimodule with $\lambda_1 g\lambda_2=\varepsilon(\lambda_1)g\varepsilon(\lambda_2)$, written as ${}_\varepsilon G_\varepsilon$. Since ${}_\varepsilon K$ is a left Λ-module, we can consider the complex

$$B(\Lambda,{}_\varepsilon K):\ \cdots\longrightarrow B_n(\Lambda,{}_\varepsilon K)\xrightarrow{\partial_n}B_{n-1}(\Lambda,{}_\varepsilon K)\longrightarrow\cdots\longrightarrow B_1(\Lambda,{}_\varepsilon K)\xrightarrow{\partial_1}B_0(\Lambda,{}_\varepsilon K)\longrightarrow 0$$

If we have to recall that $B_n(\Lambda,{}_\varepsilon K)=\Lambda\otimes_K(\Lambda/K)^n\otimes_K{}_\varepsilon K\cong\Lambda\otimes_K(\Lambda/K)^n=B_n(\Lambda)$, then we get the complex

$$B(\Lambda):\ \cdots\longrightarrow B_n(\Lambda)\xrightarrow{\partial'_n}B_{n-1}(\Lambda)\longrightarrow\cdots\longrightarrow B_1(\Lambda)\xrightarrow{\partial'_1}B_0(\Lambda)\longrightarrow 0$$

where ∂'_n is induced from ∂_n such that

$$\partial'_n(\lambda[\lambda_1|\cdots|\lambda_n])=\lambda\lambda_1(\lambda_2|\cdots|\lambda_n]$$
$$+\sum_{i=1}^{n-1}(-1)^i\ \lambda[\lambda_1|\cdots|\hat{\lambda}_i|\lambda_i\lambda_{i+1}|\cdots|\lambda_n]+(-1)^n\lambda[\lambda_1|\cdots|\lambda_{n-1}]\varepsilon(\lambda_n).$$

Since $B(\Lambda,{}_\varepsilon K)$ is a split exact sequence as K-modules, so does $B(\Lambda)$.
Under the above situation the ***reduced bar resolution*** for an augmented algebra Λ is a complex $\overline{B}(\Lambda)=K_\varepsilon\otimes_\Lambda B(\Lambda)$. The boundary ∂''_n: $\overline{B}_n(\Lambda)\longrightarrow\overline{B}_{n-1}(\Lambda)$ is defined by $\partial''_n=1_{K_\varepsilon}\otimes\partial'_n$. Also we see that $\overline{B}(\Lambda)$ is split exact sequence as K-modules and $\partial''_n\partial''_{n+1}=0$. In this

case, we have to note that

$$\bar{B}_n(\Lambda)={}_\varepsilon K_\varepsilon\otimes_\Lambda(\Lambda\otimes_K(\Lambda/K)^n)\cong K_\varepsilon\otimes_K(\Lambda/K)^n\cong(\Lambda/K)^n$$

$$\bar{B}_0(\Lambda)=K_\varepsilon\otimes_\Lambda(\Lambda\otimes_K(\Lambda/K)^0)\cong K,$$

and thus we shall put $\bar{B}_n(\Lambda)=(\Lambda/K)^n$ and $\bar{B}_0(\Lambda)=K$.

Theorem 4.3 Under the above situation, if M is a right Λ-module and G is a K-module, then we have the following isomorphisms:

$$H_n(\Lambda, {}_\varepsilon M)\cong H_n(M\otimes_\Lambda B(\Lambda)),\quad H_n(\Lambda, {}_\varepsilon G_\varepsilon)\cong H_n(G\otimes_K\bar{B}(\Lambda)).$$

Proof. Recall that $H_n(\Lambda, {}_\varepsilon M)=H_n({}_\varepsilon M\otimes_{\Lambda\text{-}\Lambda}B(\Lambda,\Lambda))$. By (V), we get the following:

$${}_\varepsilon M\otimes_{\Lambda\text{-}\Lambda}(\Lambda\otimes_K(\Lambda/K)^n\otimes_K\Lambda)\cong M\otimes_K(\Lambda/K)^n.$$

Since $M\otimes_\Lambda B(\Lambda)=M\otimes_\Lambda(\Lambda\otimes_K(\Lambda/K)^n)\cong M\otimes_K(\Lambda/K)^n$, we have the isomorphism

$$M\otimes_\Lambda B_n(\Lambda)\cong{}_\varepsilon M\otimes_{\Lambda\text{-}\Lambda}B_n(\Lambda,\Lambda)\quad(n=0,1,2,\cdots).$$

By the definition ∂ and ∂', the above isomorphism is a chain isomorphism

$$M\otimes_\Lambda B(\Lambda)\cong{}_\varepsilon M\otimes_{\Lambda\text{-}\Lambda}B(\Lambda,\Lambda).$$

Hence, we have

$$H_n(\Lambda, {}_\varepsilon M)=H_n({}_\varepsilon M\otimes_{\Lambda\text{-}\Lambda}B(\Lambda,\Lambda))\cong H_n(M\otimes_\Lambda B(\Lambda)).$$

Similarly, by (V) and our definition $\bar{B}_n(\Lambda)$, we have

$${}_\varepsilon G_\varepsilon\otimes_{\Lambda\text{-}\Lambda}(\Lambda\otimes_K(\Lambda/K)^n\otimes_K\Lambda)\cong{}_\varepsilon G_\varepsilon\otimes_K(\Lambda/K)^n=G\otimes_K\bar{B}_n(\Lambda).$$

Therefore we get the following isomorphism

$$H_n(\Lambda, {}_\varepsilon G_\varepsilon)=H_n({}_\varepsilon G_\varepsilon\otimes_{\Lambda\text{-}\Lambda}B(\Lambda,\Lambda)\cong H_n(G\otimes_K\bar{B}(\Lambda)).///$$

References

1. H. Bass: Finitistic Dimension and a Homological Generalization of semi-primary Rings. *Trans. A.M.S.* ***95*** pp.466∼488 (1960).
2. D.A. Buchsbaum: A Note on Homology in categories. *Ann. of Math.* ***69***, pp.66 ∼74, (1959).
3. H. Cartan and S. Eilenberg: Homological Algebra. Princeton, N.J., Princeton University Press (1959).
4. A. Dold: Chern Classes in General Cohomology. Symposia Mathmatica pp.385∼ 410, (1970).
5. A. Dold and D. Puppe: Homologie nicht-additives Funktoren. Anwendungen. *Ann. Inst. Fourier* ***11***, pp.201∼312, (1961).
6. P.S. Green: A Cohomology Theory based upon self-conjugacies of complex vector Bundles. *Bull. A.M.S.* ***70***, pp.522∼524, (1964).
7. D.K. Harrison: Commutative Algebras and Cohomology. *Trans. A.M.S.* ***104***, pp.191∼204, (1962).
8. P.J. Hilton: General Cohomology Theory and K-theory. New York, Cambridge University Press (1971).
9. K. Lee: Cosingular Cohomology of Coalgebras. *J. of Math. Nagoya* ***47***, pp.199 ∼225, (1972).
10. K. Lee: Singular Cohomology with Coefficients in Sheaves. *Mathematica Japonicae. Vol.* ***19***, No.3, pp.267∼274, (1974).
11. S. MacLane: Homology. Springer-Verlag (1963).
12. J. Schmid: Zu den Reduktion-esätzen in the homologi schen Theorie der Gruppen. *Archiv. der Math.* ***15***, pp.28∼32, (1964).
13. U. Shukla: Cohomolgie des algèbre associatives. *Ann. Sci. École norm. Sup.* ***78***, pp.163∼209, (1961).
14. C.T.C. Well: On the Cohomology of Certain Groups. *Proc. Camb. Phil. Soc.* ***57***, pp.731∼733, (1961).
15. G.W. Whitehead: Generalized Homology Theories. *Trns. A.M.S.* ***102***, pp.227∼ 283, (1962).

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