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Some Properties of R(A)

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Let A be a C^*-algebra. A two-sided ideal of A is said to be *primitive* if it is the kernel of a non-zero irreducible representation of A. We denote by Prim(A) the set of all primitive two-sided ideals of A. For each subset T of Prim(A), let $I(T)$ be the intersection of elements of T, which is also a two-sided ideal of A. We denote by $\bar{T}$ the set of all primitive ideals of A containing $I(T)$. Then we can prove that

(i) $\bar{\phi}=\phi$ (ϕ: null set),

(ii) $T\subset\bar{T}$ for $T\subset$Prim(A),

(iii) $T\subset$Prim(A) $\Longrightarrow$ $\bar{T}=\bar{\bar{T}}$,

(iv) T_1, $T_2\subset$Prim(A) $\Longrightarrow$ $\overline{T_1\cup T_2}=\bar{T}_1\cup\bar{T}_2$.

It follows that there is a unique topology on Prim(A) such that for each $T\subset$Prim(A) $\bar{T}$ is the closure of T in this topology. This topology is called the *Jacobson topology* on Prim(A). In this paper, we want to prove Theorem 5 which is about some properties of $R(A)$.

Definition 1. Recall that $R(A)$ is the set of all nontrivial irreducible representations of A, where A is a C^*-algebra. The *spectrum* of A is the set $R(A)$ endowed with the topology consisting of the inverse images of the Jacobson topology under the map

$$R(A)\rightarrow \text{Prim}(A)$$

which is defined as follows: For each $\pi\in R(A)$ $\pi\longmapsto\ker\pi$, where $\pi\in R(A)$(Note that for each non-trivial irreducible representation π of A in a Hilbert space $\ker\pi$ is a primitive two-sided ideal of A. In particular the map $R(A)\rightarrow$Prim(A) is surjective.

The topological space Prim(A) is a T_0-*space*. Hence the following three conditions are equivalent:

(i) $R(A)$ is a T_0-*space*,

(ii) The map ϕ: $R(A)\rightarrow$Prim(A) is a homeomorphism,

(iii) Two irreducible representations of A with the same kernel are equivalent.

Let A be a unital C^*-*algebra*, and let x be a normal element of A. we put

A' = the set of all continuous complex-valued functions defined on $\sigma_A(x)$ (where $\sigma_A(x)$ = the spectrum of x). Then there exists a unique morphism $\phi: A' \to A$ such that $\phi(1)=1$ and $\phi(\nu)=x$, where $1: A' \to C(z| \longrightarrow 1)$ and $\nu: A' \to C(z| \longrightarrow z)$. In this case $\phi(A')$ is the C^*-subalgebra of A generated by $\{1, x\}$. We shall put $\phi(f)=f(x)$ for each $f \in A'$.

Suppose two unital C^*-algebras A, B and a morphism ψ: $A \to B$ such that $\psi(1)=1$ are given and for each normal element $x \in A$ $\psi(x)$ is normal in B. Then by $\rho(x)=\|x\|$ (where $\rho(x)$ = the spectral radius of x, *i.e.*, $\rho(x)=\sup\{|\lambda| \mid \lambda \in \sigma_A(x)\}$) and $\|\psi(x)\| \leqq \|x\|$ we see that $\sigma_B(\psi(x)) \subset \sigma_A(x)$ for each normal element $x \in A$ ([4]). In this case, for a continuous complex-valued function f on $\sigma_A(x)$ if the restriction of f to $\sigma_B(\psi(x))$ is denoted by f then $\psi(f(x))=f(\psi(x))$ (※)$_1$ ([5]).

Lemma 2. Let A be a unital C^*-algebra.

(i) For an hermitian element $\alpha \in A$ and a closed set $L \subset R$, $Z=\{\pi \in R(A) \mid \sigma_{L(H_\pi)}(\pi(\alpha)) \subset L\}$ is closed in $R(A)$, where $H\pi$ is the space of π.

(ii) For each $x \in A$ the map $R(A) \to R(\pi| \longrightarrow \|\pi(x)\|)$ is a lower semicontinuous function.

Proof. (i): For each $\mu \in \bar{Z}$ assume that $\sigma_{L(H_\mu)}(\mu(\alpha))$ contains an element $\alpha \notin L$. Let f: $R \to R$ be a continuous function vanishing on L with $f(\alpha) \neq 0$. Then, for each $\pi \in Z$ $f(\alpha)=0$ $(f(\alpha) \in L(H\pi))$ since $\sigma_{L(H_\pi)}(\pi(\alpha)) \subset L$ and $f|L=0$. Hence we have

$$0=\pi(f(\alpha))=f(\pi(\alpha))$$

for $\pi \in Z$ by (※)$_1$ and $\mu(f(\alpha))=f(\mu(\alpha)) \neq 0$ since $\sigma_{L(H_\pi)}(\mu(\alpha)) \not\subset L$ and $f(\alpha) \neq 0$. This implies that μ is not a limit of any sequence $\{\pi_n\} \subset Z$. That is, $\mu \notin \bar{Z}$ which contradicts our assumption that $\mu \in \bar{Z}$.

(ii): For each $x \in A$, since $\|\pi(x)\|^2=\|\pi(x^*x)\|$ and x^*x is hermitian we can assume that $x \in A^+$. Let $k \geqq 0$ and

$$Z=\{\pi \in R(A) \mid \ \|\pi(x)\| \leqq k\}.$$

Then $Z=\{\pi \in R(A) \mid \sigma_{L(H_\pi)}(\pi(x)) \subset [0, k]\}$. By (i) Z is closed in $R(A)$. Thus

$$\{\pi \in R(A) \mid \|\pi(x)\| > k\}$$

is open in $R(A)$. That is, $\{\pi \in R(A) \mid -\|\pi(x)\| < -k\}$ is open in $R(A)$. Hence $\pi| \longrightarrow -\|\pi(x)\|$ is an upper semicontinuous function. This means that $\pi| \longrightarrow \|\pi(x)\|$ is lower semicontinuous.

Definition 3. For a C^*-algebra A, let S be a subset of $R(A)$. If every positive form on A defined by π and $\xi \in H\pi$ is a weak*-limit of states which are sums of positive forms associated with S then π is said to be ***weakly contained*** in S.

(Note that (a) a sum of positive forms associated with S is of the form $f_1+\cdots+f_n$, where f_i is defined by π_i and ξ_i such that $\pi_i \in S$ and $\xi_i \in H_{\pi_i}$, (b) f is a weak*-limit of $\{f_n\}$ if and only if for all $x \in A$ $f_n(x) \to f(x)$ in $\mathbf{C}$, where f and $f_n(n=1, 2\cdots)$ are states of A.)

Let A be a C^*-algebra, $\pi \in R(A)$ and $S \subset R(A)$. Then the following conditions are equivalent (※)$_2$.

(i) $\pi \in S$.

(ii) π is weakly contained in S.

(iii) At least one of the non-zero forms associated with π is a weak*-limit of positive form associated with S.

(iv) every state associated with π is a weak*-limit of states associated with S.

Let A be a C^*-algebra. For each subset S of $P(A)$ we put

$$Q(S)=\{f \mid \text{positive form on } A, f \text{ is a weak*-limit of sums of positive forms associated with } S\}.$$

By taking $\{Q(S) \mid S \subset P(A)\}$ as a closed base of $P(A)$ we can make $P(A)$ a topological space. In this case, we can prove that

(i) $P(A)$ is a Baire space (※)$_3$([3]),

(ii) the map $P(A) \to R(A)$ defined as in Proposition 3.4 is a continuous open map(※)$_4$ ([5]).

Lemma 4. $R(A)$ is a Baire space, where A is a C^*-algebra.

Proof. Let $\{V_1, \cdots\}$ be a decreasing sequence of dense open subsets of $R(A)$ and let U_i be the image of $V_i(i=1,2,\cdots)$ under the canonical map defined as above. Then, by Definition 3, (※)$_2$, (※)$_4$ and the topology of $P(A)$ U is open and dense in $P(A)$. By (※)$_3$ $P(A)$ is a Baire space and $\bigcap_{i=1}^{\infty} U_i$ is dense in $P(A)$, $\bigcap_{i=1}^{\infty} V_i$ is the image of $\bigcap_{i=1}^{\infty} U_i$ under the canonical map $P(A) \to R(A)$ as above, and by (※)$_2$ $\bigcap_{i=1}^{\infty} V_i$ is dense in $R(A)$. That is, $R(A)$ is a Baire space. ■

Theorem 5. Let A be a C^*-algebra.

(i) If $\{0\}$ is a primitive ideal, then $R(A)$ contains a point which is dense in $R(A)$. Furthermore, any two nonempty open subsets of $R(A)$ have non-empty intersection.

(ii) If $\{0\}$ is not a primitive ideal, and if A is separable, then $R(A)$ contains a disjoint pair of nonempty open sets.

Proof. (i): Assume that $\{0\}$ is a primitive ideal of $\mathrm{Prim}(A)$, then there exists a representation π of A such that $\ker\pi=\{0\}$. Since every primitive ideal contains $\{0\}$ we have $\overline{\{0\}}=\mathrm{Prim}(A)$. That is, $\{\pi\}$ is dense in $R(A)$. Consider two non-empty open sets O_1 and O_2 in $R(A)$. If $\pi\notin O_1(\ker\pi=\{0\})$ then $\pi\in R(A)-O_1$ and

$$R(A)-O_1=\overline{R(A)-O_1}=R(A)$$

which contradicts $R(A)-O_1\subsetneqq R(A)$. By the same reason as above $\pi\in O_2$. Therefore

$$\pi\in O_1\cap O_2\neq\phi \text{ (null set).}$$

(ii): Since A is separable there exists a sequence $\{x_n|n=1,2,\cdots\}\subset A$ such that $\overline{\{x_n|n=1,2,\cdots\}}=A$. Put $U_i=\{\pi\in R(A)|\ \|\pi(x_i)\|>1\}$,

then by(ii) of Lemma 2 U_i is open in $R(A)$. We shall prove that $\{U_n|n=1,2,\cdots\}$ is a countable open base of $R(A)$.

Let U be an open subset of $R(A)$ and let $\pi\in U$.

Then there exists a closed set N in $\mathrm{Prim}(A)$ such that $R(A)-\phi^{-1}(N)=U(\phi\colon R(A)\approx\mathrm{Prim}(A))$. In this case, by the Jacobson topology on $\mathrm{Prim}(A)$ there exists a subset T of $\mathrm{Prim}(A)$ such that $\overline{T}=N(\overline{T}=$the set of all primitive ideals containing $\bigcap_{\ker\pi'\in T}\ker\pi'$.) We put

$$I=\bigcap_{\ker\pi'\in T}\ker\pi'.$$

Then for each $\pi'\in R(A)$ with $\pi'\in\phi^{-1}(N)=\phi^{-1}(\overline{T})$ $\ker\pi'\supset I$ and for each $\pi\in R(A)$ with $\pi\notin\phi^{-1}(N)=\phi^{-1}(\overline{T})$, i. e., $\pi\in U$ $\ker\pi\not\supset I$. Hence there exists an element $y\in I$ such that $\pi(y)\neq0$. Put $x=\frac{2}{\|y\|}y\in I$ then $\|\pi(x)\|=2$ and for all $\mu\in R(A)-U$ $\mu(x)=0$. Since $\{x_n|n=1,2,\cdots\}$ is dense in A there is an element x_i such that $\|x-x_i\|<1$. Since

$$|\,\|\pi(x)\|-\|\pi(x_i)\|\,|\leqq\|\pi(x-x_i)\|\leqq\|x-x_i\|<1$$

$\|\pi(x_i)\|>1(\pi\in U_i)$ and for $\mu\in R(A)-U$ since

$$\|\mu(x-x_i)\|=\|\mu(x_i)\|\leqq\|x-x_i\|<1$$

$\mu\notin U_i$. This means that $\pi\in U_i\subset U$.

Next, taking $\pi\in R(A)$ then there exists a non-zero element $x\in\ker\pi$ because $\ker\pi\neq\{0\}$. Then there exists an element $x_i\in\{x_n|n=1,2\cdots\}$ such that $\|x-x_i\|<1$. Since

$$\|\pi(x-x_i)\|=\|\pi(x_i)\|\leqq\|x-x_i\|<1$$

$\pi\notin U_i$. By(ii) of Lemma 2

$$\pi\in\{\mu\in R(A)|\ \|\mu(x_i)\|\leqq1\}=Z_i=R(A)-U_i$$

is closed in $R(A)$. There are two cases such that

(a) there is an inner point π' in Z_i,

(b) there is not any inner point in Z_i.

In case(a), there is an open neighborhood V of π' such that $V\subset Z$. Then $V\cap U_i=\phi$(null set). In case(b), since $R(A)$ is a Baire space(by Lemma 4.)

$$\overline{R(A)-Z_i}=\overline{U}_i=R(A).$$

Therefore $\bigcap_{i=1}^{\infty} U_i$ is dense in $R(A)$ because $R(A)$ is a Baire space.

Suppose $\eta\in\bigcap_{i=1}^{\infty}U_i$. Then, for each $x_i\in\{x_n|n=1,2,\cdots\}$ $\|\eta(x_i)\|>1$. For $x_j\in\{x_n|n=1,2\cdots\}$ such that $\|0-x_j\|<1$ $\|\eta(x_j)\|\leq\|0-x_j\|=\|x_j\|<1$, which implies that $\bigcap_{i=1}^{\infty} U_i$ is an empty set. This contradicts that $\bigcap_{i=1}^{\infty} U_i$ is dense in $R(A)$. Hence the case (b) does not happen in $R(A)$. ■

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