HONAM MATHEMATICAL JOURNAL
Volume 6, Number 1, August. 1984

A Simply Acting Subgroup of the Fundamental Group

By
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1. Introduction.

In the study of fixed point properties of a continuous self-map $f: X \to X$ on a connected compact ANR's(absolute neighborhood retracts), several interesting numbers are associated with f. These are the Reidemeister number of f, denoted by $R(f)$, the Lefschetz number of f, denoted by $L(f)$, and the Nielsen number of f, denoted by $N(f)$ (cf. §2). In particular, we are interested in the $N(f)$ which is not only a lower bound for the cardinality of the set of fixed points of f but also a homotopy invariant. S. Lefschetz showed that if $L(f) \neq 0$ then any map g homotopic to f must have a fixed point on X ([13]). J.B. Jiang introduced a Jiang subgroup $T(f, x_0)$ for any map f on X. This $T(f, x_0)$ is a subgroup of the fundamental group $\Pi_1(X, f(x_0))$ of X. Jiang proved that when $T(f, x_0) = \Pi_1(X, f(x_0))$, every fixed point class has same index k for some integer $k(X, f)$, and $L(f) = k\ N(f)$. In particular $N(f) = 0$ if $L(f) = 0$, and $N(f) = R(f)$ if $L(f) \neq 0$([9]).

While Jiang condition $(T(f, x_0) = \Pi_1(X, f(x_0)))$ guarantees an easy calculation for $N(f)$, only limited spaces satisfy the Jiang condition. If we can weaken the requirement of Jiang and still get useful results for $N(f)$, or analyze spaces that satisfy the Jiang condition this would contribute to many areas of mathematics.

Here the author's goal is to improve this Jiang condition not to have such a strong restriction on space X yet still be able to calculate this $N(f)$ easily and also try to analyze spaces that satisfy the Jiang condition, where the lack of knowledge of $N(f)$ is at present a formidable obstacle to the solutions of many important open problems. At first, the author defines a new subgroup $P(X, f, x_0)$ of $\Pi_1(X, f(x_0))$ such that $T(f, x_0) \subseteq P(X, f, x_0)$ (Definition 4.1), and show that it is a homotopy invariant of f(Theorem 4.1). Secondly, the author gets a condition to be $P(X, f, x_0) = \Pi_1(X, f(x_0))$ (Theorem 4.2), and can weaken the Jiang condition by putting $P(X, f, x_0) = \Pi_1(X, f(x_0))$. Thus the author shows that if X is connected aspherical and $P(X, f, x_0) = \Pi_1(X, f(x_0))$, then $T(f, x_0) = \Pi_1(X, f(x_0))$ and every nice properties of Jiang's follows (Theorem 4.3 and corollary 4.5-Corollary 4.7). Lastly the relationship of $P(X, x_0)$ and $P(X, f, x_0)$ is considered (Theorem 4.10).

2. Preliminaries (Outline of fixed point theory)

We will introduce some terminology and notation and will summarize those definitions and results that will be used from here on from [11].

We always assume X to be a connected compact ANR (absolute neighborhood rectract) with x_0 as a base point. It is well known that X has a universal covering space. Let $p:\tilde{X}\longrightarrow X$ be the universal covering of X. A lifting of a map $f:X\longrightarrow X$ is a map $\tilde{f}:\tilde{X}\longrightarrow \tilde{X}$ such that $p\circ\tilde{f}=f\circ p$. A covering transformation is a map $r:\tilde{X}\longrightarrow\tilde{X}$ such that $p\circ r=p$, that is, a lifting of the identity map $1_X:X\longrightarrow X$. The set of all covering transformations form a group, we denote it by $G(\tilde{X}, X)$ which is isomorphic to $\Pi_1(X, x_0)$ (Sometimes we will write $\Pi_1(X)$ rather than $\Pi_1(X, x_0)$ for brevity). The set of all fixed points of $f:X\longrightarrow X$ we will denote by Fix(f). Two liftings $\tilde{f}$ and $\tilde{f}'$ of $f:X\longrightarrow X$ are said to be *conjugate* if there exists $r\in G(\tilde{X}, X)$ such that $\tilde{f}'=r\circ\tilde{f}\ r^{-1}$. The *lifting classes* are the equivalence classes by conjugacy. A lifting class containing $\tilde{f}$ is denoted by $[\tilde{f}]$, that is,

$$[\tilde{f}]=\{r\circ\tilde{f}\circ r^{-1}\,|\,r\in G(\tilde{X}, X)\}.$$

The following results are well known:

(i) $\mathrm{Fix}(f)=\bigcup_{\tilde{f}} p\cdot\mathrm{Fix}(\tilde{f})$

(ii) $p\,\mathrm{Fix}(\tilde{f})=p\,\mathrm{Fix}(\tilde{f}')$, if $[\tilde{f}]=[\tilde{f}']$

(iii) $p\,\mathrm{Fix}(\tilde{f})\cap p\,\mathrm{Fix}(\tilde{f}')=\phi$, if $[\tilde{f}]\neq[\tilde{f}']$ ([11], p.5)

The subset p Fix $(\tilde{f})$ of Fix(f) is called the *fixed point class of f* determined by the lifting class$[\tilde{f}]$. Thus the fixed point set Fix(f) is splited into a disjoint union of fixed point classes. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number of f*, denoted by $R(f)$. It is a positive integer or infinity. Here we introduce two results for the fixed point classes:

(1) Every fixed point class of $f:X\longrightarrow X$ is an open subset of Fix(f).

(2) Every map $f:X\longrightarrow X$ has only finitely many non-empty fixed point classes, each a compact subset of X([11], p.7).

Now we will introduce the definition of the fixed point index of a self-map([11], p.13 ~17).

Let U be an open subset of the Euclidean n-space R^n and $f:U \longrightarrow R^n$ be a given map. Suppose that $a \in U$ is an isolated fixed point of f. Pick a sphere S_a^{n-1} centered at a, small enough to exclude the other fixed points. For any $x \in S_a^{n-1}$ the vector $x-f(x)$ is not zero, so a map $\varphi: S_a^{n-1} \longrightarrow S^{n-1}$ is defined by taking

$$\varphi(x) = \frac{x-f(x)}{|x-f(x)|}.$$

φ is called a direction field. The fixed point index of f at a is defined by

$$\text{index}(f,a) = \text{degree of } \varphi.$$

Suppose that for a map $f:U \longrightarrow R^n$ all the fixed points $\{a_j\}$ of f are isolated. Pick a small $S_{a_j}^{n-1}$ for each a_j and define the direction field $\varphi: U-\text{Fix}(f) \longrightarrow S^{n-1}$ as before. Suppose that $i_j = \text{degree}\,(\varphi | S_{a_j}^{n-1})$, then $i_j = \text{index}(f,a_j)$. We define

$$\text{index}(f,U) = \sum_j i_j = \sum_j \text{index}\,(f,a_j).$$

This number has the alternative definition. Take an open subset V of U such that $\bar{V} \subset U$, V contains all the fixed points of f, and $\bar{V}$ is a smooth n-mainfold in U. Now φ is defined on ∂V, so we have degree $(\varphi) = i$ and this i is the sum of those i_j's.

Suppose that for a map $f:U \longrightarrow R^n$, $\text{Fix}\,(f)$ is compact. Take any open set $V \subset U$ such that $\text{Fix}\,(f) \subset V \subset \bar{V} \subset U$ and $\bar{V}$ is a smooth n-mainfold, then φ is defined on ∂V and we have $\text{degree}(\varphi) = i$. This i is defined to be the fixed point index of f on U, denoted by $\text{index}(f,U)$.

Here we introduce the definition of the fixed point index for a map on the connected compact ANR X. Every connected compact ANR can be embedded in some Euclidean space as a neighborhood retract. Let U be an open subset of X and $f:U \longrightarrow X$ be a given map. X can be embedded in R^N with inclusion $i:X \longrightarrow R^N$. There in a neighborhood W of $i(X)$ in R^N and a retraction $r:W \longrightarrow X$ such that $r \circ i = 1_X$. We have a diagram

$$\begin{array}{ccccc} X & \supset & U & \xrightarrow{\;f\;} & X \\ r\uparrow & & r\uparrow & i\circ f\circ r & \downarrow i \\ R^N \supset W & \supset & r^{-1}(U) & \longrightarrow & R^N \end{array}$$

when $\text{Fix}(f)$ is compact, define the fixed point index to be

$$\text{index}(f,U) = \text{index}(i\circ f\circ r,\ r^{-1}(U)),$$

the latter being the index in R^N, $\text{index}(f,U)$ is called the fixed point index of f in U.

The basic facts about the fixed point index are listed below.

Let X be a connected compact ANR, $U \subset X$ an open subset, $f:U \longrightarrow X$ a map such that

Fix(f) is compact. The index(f,U) has the following properties:

(i) Existence of fixed points. If index(f,U)$\neq 0$, then f has at least one fixed point in U.

(ii) Homotopy invariance. If $H=\{h_t\}:f_0\simeq f_1:U\longrightarrow X$ is a homotopty such that $\bigcup_{t\in I}$ Fix(h_t) is compact, then

$$\text{index}(f_0, U)=\text{index}(f_1, U).$$

(iii) Additivity. Suppose $U_1,\cdots\cdots, U_s$ are disjoint open subsets of U, and f has no fixed point on $U-\bigcup_{j=1}^{s} U_j$. If index(f,U) is defined, then index(f,U_j), ($j=1,\cdots\cdots,s$) are all defined and

$$\text{index}(f,U)=\sum_{j=1}^{s}\text{index}(f,U_j),$$

where index(f,U_j)=index($f|U_j$, U_j).

(iv) Multiplicativity. Let X, Y be connected compact ANRs, $U\subset X$ and $V\subset Y$ be open subsets, $f:U\longrightarrow X$ and $g:V\longrightarrow Y$ be maps. Consider the product $f\times g$: $U\times V\longrightarrow X\times Y$. If index($f,U$) and index($g,V$) are defined, so is index($f\times g$, $U\times V$), and

$$\text{index}(f\times g,\ U\times V)=\text{index}(f,\ U)\cdot\text{index}(g,\ V).$$

(v) Commutativity. Let $f:U\longrightarrow Y$ and $g:V\longrightarrow X$ be maps. The composite maps $g\circ f$: $f^{-1}(V)\longrightarrow X$ and $f\circ g:g^{-1}(U)\longrightarrow Y$ are defined, and f,g restrict to a pair of homeomorphisms between the fixed point sets

$$\text{Fix}(g\circ f)\ \underset{g}{\overset{f}{\rightleftarrows}}\ \text{Fix}(f\circ g).$$

If index($g\circ f$, $f^{-1}(V)$) is defined, so is index($f\circ g$, $g^{-1}(U)$), and

$$\text{index}(g\circ f,\ f^{-1}(V))=\text{index}(f\circ g,\ g^{-1}(U)).$$

(vi) Normalization. If $f:X\longrightarrow X$, then

$$\text{index}(f,X)=L(f)=\sum_q(-1)^q\ \text{trace}(f_{q*}:H_q(X;Q)\longrightarrow H_q(X;Q)),$$

where Q is the field of rational numbers. The number $L(f)$ is called the *Lefschetz number of f.*

(vii) Removability condition. Suppose a map $f:U\longrightarrow X$ has only one fixed point x_0, with index (f,U)=0, and x_0 has a Euclidean neighborhood. Then, given any neighborhood V of x_0, there is a map $g\simeq f:U\longrightarrow X$ such that $g=f$ on $U-V$ and g has no fixed point on U([11], p. 13~17).

Let $U\subset X$ be an open set and $f:U\longrightarrow X$ a map such that Fix(f) is compact. A set of fixed points $S \subset \mathrm{Fix}(f)$ is called an *isolated set of fixed points* if S is compact and open in $\mathrm{Fix}(f)$, that is, if both S and $\mathrm{Fix}(f)-S$ are compact. The index of an isolated set S of fixed points, denoted by $\mathrm{index}(f,S)$, is defined as follows. Pick a neighborhood $W \subset U$ of S isolating S from other fixed points, that is, such that $S=W \cap \mathrm{Fix}(f)$. Define

$$\mathrm{index}(f,S)=\mathrm{index}(f,W).$$

Let F be a fixed point class of $f:X \longrightarrow X$. F is an isolated set of fixed points by (1). So, $\mathrm{index}(f,F)$ is defined. F is *essential* if $\mathrm{index}(f,F)\neq 0$, *inessential* if $\mathrm{index}(f,F)=0$. The number of essential fixed point classes of f is called the *Nielsen number of f*, denoted $N(f)$. We have the following results (by the definitions):

(i) $N(f) \leq R(f)$

(ii) Each essential fixed point class is non-empty.

(iii) $N(f)$ is a non-negative integer, $0 \leq N(f) < \infty$.

(iv) $N(f) \leq$ Cardinal number of $\mathrm{Fix}(f)$.

We need more terminologies and notations. For a while, let X be any space. For each $n>0$, every path $\sigma:I \longrightarrow X$ gives in a natural way an isomorphism

$$\sigma_n:\Pi_n(X,x_1) \longrightarrow \Pi_n(X,x_0),\ x_0=\sigma(0),\ x_1=\sigma(1),$$

which depends only on the homotopy class of the path σ (relative to end points). If σ is the constant path $\sigma(I)=x_0$, then σ_n is the identity automorphism. As an immediate consequence of the above fact, we know that the fundamental group $\Pi_1(X,x_0)$ operates on $\Pi_n(X,x_0)$, $n \geq 1$, as a group of automorphisms. Since σ_n is an isomorphism, for a path-connected space X, all the groups $\Pi_n(X,x_0)$ with various basic points $x_0 \in X$ are isomorphic. Hence, as an abstract group, $\Pi_n(X,x_0)$ does not depend on the basic point x_0 and sometimes we will write $\Pi_n(X)$ rather than $\Pi_n(X,x_0)$ for brevity. In the special case $n=1$, we can see that, for any two elements g and h in $\Pi_1(X,x_0)$, h operates on g as follows:

$$hg=hgh^{-1} \qquad ([8],\ \text{p.}130)$$

Hence, if $\Pi_1(X)$ is abelian, $hg=g$ for every $g \in \Pi_1(X)$ and $h \in \Pi_1(X)$. Let W be a group which operates on a group G. We shall say that W operates (or acts) simply (or trivially) on G if $wg=g$ for every $w \in W$ and $g \in G$. For a given integer $n>0$, a path-connected space X is *n-simple* if there exists a point $x_0 \in X$ such that $\Pi_1(X,x_0)$ operates simply on $\Pi_n(X,x_0)$. By the above fact, a path-onnected space X is 1-simple if and only if $\Pi_1(X)$ is abelian. We say that $\Pi_n(X,x_0)$ is *simple* if $\Pi_1(X)$ operates simply on $\Pi_n(X,x_0)$.

3. Jiang subgroup of a self-map

Let X be a topological space with x_0 as a base point. A homotopy h_t: $X \longrightarrow X$ is called a *cyclic homotopy based at* 1_X (identity map of X), if $h_0 = h_1 = 1_X$. For a given self-map $f: X \longrightarrow X$, a homotopy $h_t: X \longrightarrow X$ is called a *cyclic homotopy based at* f if $h_0 = h_1 = f$. If h_t is a cyclic homotopy, the path given by $\sigma: \mathrm{I} \longrightarrow X$ such that $\sigma(t) = h_t(x_0)$ will be called the *trace* of h_t. The trace is obviously a closed path.

Definition 3.1. The set of homotopy classes of those loops which are the trace of some cyclic homotopy at 1_X form a subgroup of the fundamental group $\Pi_1(X, x_0)$ which we shall denote by $G(X, x_0)$ or simply by $G(X)$, that is,

$$G(X, x_0) = \{\alpha \in \Pi_1(X, x_0) \mid \text{there exists a cyclic homotopy } h_t: 1_X \simeq 1_x \text{ such that } [h_t(x_0)] = \alpha\}. \quad ([7], \text{ p.840}).$$

Definition 3.2. The another trace subgroup of cyclic homotopies at $f: X \longrightarrow X$, denoted by $T(f, x_0)$ or simply by $T(f)$ ($\subset \Pi_1(X, f(x_0))$) is defined by

$$T(f, x_0) = \{\alpha \in \Pi_1(X, f(x_0)) \mid \text{there exists a cyclic homotopy } h_t: f \simeq f \text{ such that } [h_t(x_0)] = \alpha\}.$$

$G(X, x_0)$ is exactly the same as $T(1_X, x_0)$. If $T(1_X, x_0)$ is abbreviated by $T(X)$, then we have

$$T(X) = T(1_X, x_0) = G(X, x_0).$$

$T(f, x_0)$ is called the Jiang subgroup of f, is indeed a subgroup of $\Pi_1(X, f(x_0))$ ([5], p.97 and [11], p.31).

We have mentioned that the elements of $\Pi_1(X, x_0)$ operate on $\Pi_n(X, x_0)$ as a group of automorphisms. From now on we shall concern ourselves only with connected compact ANRs in this paper. Let X be one such with x_0 as a base point.

Definition 3.3. The set of elements of $\Pi_1(X, x_0)$ which operate trivially on $\Pi_n(X, x_0)$ for $n \geq 1$ form a subgroup which will be denoted as $P(X, x_0)$. ([7], p.843).

The subgroup of $\Pi_1(X, x_0)$ which operate trivially on $\Pi_1(X, x_0)$ itself is precisely the center of $\Pi_1(X)$, hereafter denoted by $Z(\Pi_1(X))$. Thus we have $P(X, x_0) \subseteq Z(\Pi_1(X))$. By Gottlieb, we have

$$G(X, x_0) \subseteq P(X, x_0) \quad ([7], \text{ p.843}).$$

So we have

$$G(X,x_0)\subseteq P(X,x_0)\subseteq Z(\Pi_1(X))\subseteq \Pi_1(X,x_0).$$

By Brown, we also have

$$G(X,x_0)\subseteq T(f,x_0) \ ([5]).$$

Here the author tries to see if $T(f,x_0)$ is related to $P(X,x_0)$.

Proposition 3.1. $T(f,x_0)$ is not included in $P(X,x_0)$, that is, $T(f,x_0)\subsetneq P(X, x_0)$.
Proof. Suppose that $T(f,x_0)\subseteq P(X,x_0)$ is true. Then by the fact, $P(X,x_0)\subseteq Z(\Pi_1(X))$, we have $T(f,x_0)\subseteq Z(\Pi_1(X))$, for any map $f:X\longrightarrow X$. On the other hand, if a space X has a non-abelian fundamental group, then we have $Z(\Pi_1(X))\subsetneq \Pi_1(X)$. Therefore for a map $f:X\longrightarrow X$ which satiesfies the Jiang condition, that is, $T(f,x_0)=\Pi_1(X)$, $T(f)\subseteq Z(\Pi_1(X))$ is impossible (cf. counter example below)

Counter example. Let X be a Klein Bottle. Then $\Pi_1(X)$ has the presentation: two generators α,β and one relation $\alpha\beta\alpha\beta^{-1}=e$ that is, $\Pi_1(X)=\{\alpha,\beta\,|\,\alpha\beta\alpha=\beta\}$ ([14], p.106). Thus we have $Z(\Pi_1(X))\subsetneq \Pi_1(X)$. On the other hand, we get $T(f,x_0)=\Pi_1(X)$ for a special map $f:X\longrightarrow X$ which R.F. Brown constructed ([5], p.108), and so $T(f,x_0)\subsetneq Z(\Pi_1(X))$.

In fact, in the case that X is *aspherical* (in the sense of $\Pi_n(X)=0$ for $n>1$), we have

$$T(f,x_0)=Z(f_*(\Pi_1(X)),\ \Pi_1(X))\supseteq Z(\Pi_1(X)),$$

where $Z(f_*(\Pi_1(X)), \Pi_1(X))$ means centralizer of $f_*(\Pi_1(X))$ in $\Pi_1(X)$. For the proof of the above fact, we consider the two parts, ($\supseteq$) part and ($\subseteq$) part. We can refer([5], p. 102, Theorem 10) for the ($\supseteq$) part, and ([1], p.20, Cor. Ⅳ3) for the ($\subseteq$) part.

Proposition 3.2. $P(X,x_0)$ is not included in $T(f,x_0)$, that is, $P(X,x_0)\subsetneq(T(f, x_0)$.
Proof. T.Ganea constructed that there exists a space X such that $G(X,x_0)\subsetneq P(X,x_0)$ ([6]). For our purpose, put $f=1_X$. Then we have $G(X,x_0)=T(1_X,x_0)$. Hence we have $T(1_X,x_0)\subsetneq P(X, x_0)$, that is, $P(X,x_0)\subsetneq T(f,x_0)$.

Consequently, for any map $f:X\longrightarrow X$, $T(f,x_0)$ has no any relation to $P(X,x_0)$.

4. A simply acting subgroup of the fundamental group

W. Barnier showed ([1], Theorem Ⅲ.2) that any element of $T(f,x_0)$ operates trivially on $f_*\ \Pi_n(X,x_0)$, for $n\geq 1$, where f_* is the induced homomorphism by f on the homotopy groups,

From the view point of the above fact, the author defines a new collection $P(X,f,x_0)$:

Definition 4.1. The set of elements of $\Pi_1(X, f(x_0))$ which operate trivially on $f_* \Pi_n(X, x_0)$ for all $n \geq 1$ will be denoted as $P(X, f, x_0)$, that is,

$$P(X,f,x_0)=\{\alpha\in\Pi_1(X,f(x_0))\,|\,\alpha \text{ acts trivially on } f_*\,\Pi_n(X,x_0),\ \forall\ n\geq 1\}.$$

Obviously $P(X, f, x_0)$ is a subgroup of $\Pi_1(X, f(x_0))$.

When $n=1$, we can see that

$$P(X,f,x_0)\subseteq Z(f_*(\Pi_1(X)),\ \Pi_1(X,f(x_0))),$$

and so we have

$$T(f,x_0)\subseteq P(X,f,x_0)\subseteq Z(f_*(\Pi_1(X)),\ \Pi_1(X,f(x_0)))\subseteq \Pi_1(X,f(x_0)), \quad \text{(A)}$$

Theorem 4.1. If the maps f, g: $X \longrightarrow X$ are homotopic, then $P(X, f, x_0) = P(X, g, x_0)$.
Proof. Since X is a connected compact ANR, X is path-connected. So $\Pi_1(X, f(x_0))$ is isomorphic to $\Pi_1(X, x_0)$. Without loss of generality, we can assume that x_0 is the fixed base point of X under the homotopy $f \simeq g$. Thus we have $f(x_0) = x_0 = g(x_0)$ and $f_* = g_*$: $\Pi_n(X, x_0) \longrightarrow \Pi_n(X, x_0)$ for all $n \geq 1$. Therefore $f_* \Pi_n(X, x_0) = g_* \Pi_n(X, x_0)$ for all $n \geq 1$. On the other hand, since

$$P(X,f,x_0)=\{\alpha\in\Pi_1(X,f(x_0))\,|\,\alpha \text{ acts trivially on } f_*\,\Pi_n(X,x_0),\ \forall n\geq 1\}$$

and $$P(X,g,x_0)=\{\beta\in\Pi_1(X,g(x_0))\,|\,\beta \text{ acts trivially on } g_*\,\Pi_n(X,x_0),\ \forall n\geq 1\},$$

by the above results, we have $P(X, f, x_0) = P(X, g, x_0)$. Thus, $P(X, f, x_0)$ is a homotopy invariant of f.

In [9], Jiang proved that if a map $f: X \longrightarrow X$ satisfies the Jiang condition (i.e. $T(f, x_0) = \Pi_1(X, f(x_0))$), then $L(f) = kN(f)$, where k is the index of the essential fixed point classes of f. Furtheremore, if $L(f)=0$ then $N(f)=0$ and if $L(f)\neq 0$ then $N(f)=R(f)$, that is, Jiang condition assures us a lot of good properties in dealing with numbers of fixed points of $f: X \longrightarrow X$. Now what the author is interested is to weaken the Jiang condition by putting $P(X, f, x_0) = \Pi_1(X, f(x_0))$.

If f_* is an epimorphism then $f_* \Pi_n(X, x_0) = \Pi_n(X, x_0)$ for all $n \geq 1$. Thus we have $P(X, f, x_0) = P(X, x_0)$ by the definitions.

Theorem 4.2. If $\Pi_n(X, x_0)$ is simple for all $n \geq 1$ and $\alpha_n f_* = f_* \alpha_n$ for any $\alpha \in \Pi_1(X)$, then we have $P(X, f, x_0) = \Pi_1(X, f(x_0))$.

Proof. By the definitions we have that $P(X, f, x_0) = \Pi_1(X)$ if and only if $f_* \Pi_n(X, x_0)$ is simple for all $n \geq 1$, where $f_* \Pi_n(X, x_0) = \{[f\sigma] \,|\, \sigma \in \Pi_n(X, x_0)\}$. If $f_* \Pi_n(X, x_0)$ is simple then $\alpha[f\sigma]=[f\sigma]$ for any $\alpha\in P(X,f,x_0)$. If $\Pi_n(X,x_0)$ is simple then $\alpha[\sigma]=\alpha_n\sigma=\sigma$ for any $\sigma\in\Pi_n(X)$ and $\alpha\in\Pi_1(X)$. Futhermore if $f_*\ \alpha_n=\alpha_n\ f_*$ then for any $f_*\ \sigma=[f\sigma]\in f_*\ \Pi_n(X,x_0)$, we have

$$\alpha\cdot f_*\,\sigma=\alpha[f\sigma]=\alpha_n[f\sigma]=\alpha_n\,f_*(\sigma)=f_*\,\alpha_n(\sigma)=f_*\,\sigma\ \text{(since } \alpha_n\,\sigma=\sigma\text{)}.$$

Thus we have if α fix σ,

then α would also fix $f_*\,\sigma$ if and only if $f_*\,\alpha_n=\alpha_n\,f_*$

for any $\alpha\in\Pi_1(X)$, that is, if $\Pi_n(X,x_0)$ is simple

then $f_*\,\Pi_n(X,x_0)$ is simple iff $f_*\,\alpha_n=\alpha_n\,f_*$ for any $\alpha\in\Pi_1(X,x_0)$.

On the other hand the converse (if $f_*\,\Pi_n(X,x_0)$ is simple then $\Pi_n(X,x_0)$ is simple) is in generally not true. For example, in the case, f_* is a trivial map and $\Pi_n(X,x_0)$ is not simple, the converse is not true.

By R. Brown, we have the following three results, we label them as lemmas, and we will use them as lemmas hereafter.

Lemma 4.1. Let $f:X\longrightarrow X$ be a map such that $T(f,x_0)=\Pi_1(X,x_0)$, then all the fixed point classes of f have the same index. ([5], Theorem 4)

Lemma 4.2. If X is a connected aspherical (i.e. $\Pi_n(X)=0$ for all $n>1$) and $f:X\longrightarrow X$ is a map, then

$$Z(f_*(\Pi_1(X)),\ \Pi_1(X,f(x_0))\subseteq T(f,x_0)\ \text{([5], Theorem 10)}.$$

Lemma 4.3. Suppose that $f:X\longrightarrow X$ is a map such that $T(f,x_0)=\Pi_1(X,x_0)$. If $L(f)=0$, then $N(f)=0$. ([5], Cor. 5).

Theorem 4.3. Suppose that X is conected aspherical. If $P(X,f,x_0)=\Pi_1(X)$, then we have $T(f,x_0)=\Pi_1(X,x_0)$.

Proof. Recall (A):

$$T(f)\subseteq P(X,f,x_0)\subseteq Z(f_*(\Pi_1(X)),\ \Pi_1(X))\subseteq\Pi_1(X).$$

Therefore, if $P(X,f,x_0)=\Pi_1(X)$, then we have

$$Z(f_*(\Pi_1(X)),\ \Pi_1(X))=\Pi_1(X).$$

By (A) and lemma 4.2, we have $T(f)=Z(f_*(\Pi_1(X)),\ \Pi_1(X))$, and so $T(f)=\Pi_1(X,x_0)$.

By the lemmas 4.1 and 4.3, we have the following Corollaries:

Corollary 4.4. If X is connected aspherical and $P(X,f,x_0)=\Pi_1(X)$, then all the fixed point classes of $f:X\to X$ have the same index.

Corollary 4.5. Suppose that X is connected aspherical and $P(X,f,x_0)=\Pi_1(X)$. If $L(f)=0$, then $N(f)=0$.

We have that $Z(f_*\Pi_1(X), \Pi_1(X, f(x_0)))=\Pi_1(X, f(x_0))$ if and only if every element $\alpha\in\Pi_1(X)$ commutes with $[f\sigma]\in f_*\Pi_1(X)$, and every element $\alpha\in\Pi_1(X)$ commutes with $[f\sigma]\in f_*\Pi_1(X)$ if and only if $f_*\Pi_1(X)\subseteq Z(\Pi_1(X))$. Since $P(X,f,x_0)=\Pi_1(X)$ forces $Z(f_*\Pi_1(X), \Pi_1(X))=\Pi_1(X)$, for a general space X, the above says that $f_*\Pi_1(X)\subset Z(\Pi_1(X))$, so $f_*\Pi_1(X)$ is abelian.

Definition 4.2. We say $f:X\to X$ is 1-*simple* if the image of $f_*:\Pi_1(X,x_0)\longrightarrow\Pi_1(X,f(x_0))$ is contained in the center of $\Pi_1(X,f(x_0))$ i.e. $f_*\Pi_1(X)\subset Z(\Pi_1(X))$. This property of f is independent of the choice of x_0, since it is a homotopic invariant property of f. Note that if the identity map 1_X is 1-simple, i.e. if $\Pi_1(X)$ is an abelian group, then any $f:X\longrightarrow X$ is 1-simple.

Definition 4.3. For a map $f:X\to X$, two elements $\alpha,\beta\in\Pi_1(X,f(x_0))$ are said to be f-congruent, written by $\alpha\underset{R}{\sim}\beta$, if there is a $\gamma\in\Pi_1(X,x_0)$ such that $f_*(\gamma)\alpha=\beta\cdot\gamma$.

This is an equivalence relation on $\Pi_1(X,f(x_0))$; the class containing α is denoted by α_R. The set of such clesses is denoted by $\Pi_1(X,f(x_0))/R$.

Let $T(f,x_0)/R$ be the set of all calsses $\tau_R\in\Pi(X,f(x_0))/R$ with a representative in $T(f,x_0)$, and $J(f)$ is defined to be the cardinality of $T(f,x_0)/R$ ([3] p.558).

By the above fact, we have the following:

Corollary 4.6. If $P(X,f,x_0)=\Pi_1(X, f(x_0))$, then f is 1-simple.

By the result of Brooks and Brown ([3], Theorem 3 and 4, pp.560~561), we have the following:

Corollary 4.7. If $P(X,f,x_0)=\Pi_1(X, f(x_0))$, then

(i) $J(f)$ divides $R(f)$ (ii) $J(f)$ divides $N(f)$ (iii) $J(f)$ divides $L(f)$.

Thus we have the following:

Theorem 4.8. For a general space X, if $P(X,f,x_0)=\Pi_1(X,f(x_0))$, then $f_*\Pi_1(X)$ is abelian, hence we have that

(1) if $R(f)$ is prime and $J(f)\gtrsim 1$ then $N(f)=R(f)$

(2) if $L(f)$ is prime and $J(f)\gtrsim 1$ then $|L(f)|\leq N(f)$.

In the case that X is connected aspherical, we showed that under the condition $P(X, f,x_0)=\Pi_1(X)$, if $L(f)=0$ then $N(f)=0$ (Cor. 4.5). Now we will investigate this problem in the more general situation. A part of this problem is answered by Jiang ([9], Theorem

5.5), and it is as follows:

Suppose that X is a connected compact ANR such that $\Pi_1(X)$ is finite and the operation of $\Pi_1(X)$ on the isomorphic groups with rational coefficients(i.e. $H_*(\tilde{X},Q)\longrightarrow H_*(\tilde{X},Q)$) is usual. If f is 1-simple then we have

$$N(f)=|\mathrm{Coker}(1-f_{1*})| \text{ if } L(f)\neq 0$$

and $$N(f)=0 \quad \text{if } L(f)=0,$$

where f_{1*} is the induced homomorphism $H_1(X)\longrightarrow H_1(X)$.

By corollary 4.6, if $P(X,f,x_0)=\Pi_1(X)$ then f is 1-simple. When $\Pi_1(X)$ is abelian, $H_1(X)=\Pi_1(X)$ and $|\mathrm{coker}(1-f_{1*})|=|\mathrm{Coker}(1-f_*)|$, where $f_*:\Pi_1(X)\longrightarrow\Pi_1(X)$. R. Brown showed when $\Pi_1(X)$ is abelian, $R(f)=|\mathrm{Coker}(1-f_*)|$ ([5]). So we have $|\mathrm{Coker}(1-f_{1*})|=R(f)$ if $\Pi_1(X)$ is abelian.

Thus we have a following result:

Theorem 4.9. Suppose that X is a connected compact ANR such that $\Pi_1(X)$ is finite, abelian and the operation of $\Pi_1(X)$ on the isomorphic groups with rational coefficients is usual. If $P(X,f,x_0)=\Pi_1(X,f(x_0))$ then we have.

$$N(f)=R(f) \quad \text{if } L(f)\neq 0$$

and $$N(f)=0 \quad \text{if } L(f)=0.$$

By the way, we consider the relationship of $P(X,x_0)$ and $P(X,f,x_0)$, and get a following result:

Theorem 4.10. If $f:X\longrightarrow X$ is a map such that $f_*\beta_n=\beta_n f_*$ for every $\beta\in P(X,x_0)$, then $P(X,x_0)\subseteq P(X,f,x_0)$.

Proof. We recall the definitions of $P(X,x_0)$ and $P(X,f,x_0)$. $P(X,x_0)=\{\beta\in\Pi_1(X)\,|\,\beta$ acts trivially on $\Pi_n(X,x_0)$, $\forall n\geq 1\}$, $P(X,f,x_0)=\{\alpha\in\Pi_1(X)\,|\,\alpha$ acts trivially on $f_*\Pi_n(X,x_0)$, $\forall n\geq 1\}$. For any $\beta\in P(X,x_0)$ and $\sigma\in\Pi_n(X,x_0)$, $\beta\sigma=\beta_n\sigma=\sigma$. If $f_*\beta_n=\beta_n f_*$ then $\beta[f\sigma]=\beta_n f_*\sigma=f_*\beta_n\sigma=f_*\sigma=[f\sigma]$. Thus we have $P(X,x_0)\subseteq P(X,f,x_0)$.

On the other hand, if $\alpha[f\sigma]=[f\sigma]$ then it is not necessarily followed that $\alpha\sigma=\sigma$. So $P(X,f,x_0)\subsetneq P(X,x_0)$. Here we have two long arrays:

(1) If $f_*\beta_n=\beta_n f_*$ for any $\beta\in P(X,x_0)\subset\Pi_1(X)$, then we have
$T(X)\subseteq P(X,x_0)\subseteq P(X,f,x_0)\subseteq Z(f_*\Pi_1(X),\ \Pi_1(X))\subseteq\Pi_1(X)$,

(2) for any map $f:X\longrightarrow X$, we have
$T(X)\subseteq T(f,x_0)\subseteq P(X,f,x_0)\subseteq Z(f_*\Pi_1(X),\ \Pi_1(X))\subseteq\Pi_1(X)$.

References

1. W. Barnier. The Jiang subgroup for a map, Doctoral Dissertation, Univ. of California, Los Angeles, 1967.

2. R. Brooks. Coincidences, roots and fixed points, Doctoral Dissertation, Univ. of California, Los Angeles, 1967.

3. R. Brooks, and R. Brown. A lower bound for the $\triangle$—Nielsen number, *Trans. AMS* ***143***(1969), 555—564.

4. R.F. Brown. The Nielsen number of a fiber map, *Annals of Math.* ***85*** (1967), 483—493; Corrections, *Annals of Math.* ***95***(1972), 365—367.

5. __________. The Lefschetz Fixed Point Theorem, Scott-Foresman, Chicago, 1971.

6. T. Ganea, Cyclic Homotopies, *Illinois J. of Math. Vol.* ***12*** (1968).

7. D.H. Gottlieb. A certain subgroup of the fundamental group, *Amer. J. Math.* ***87*** (1965), 840—856.

8. S.T. Hu. Homotopy Theory, Academic Press Inc. 1959.

9. B.J. Jiang. Estimation of the Nielsen numbers, *Acta Math. Sinica* ***14*** (1964), 304—312(=*Chinese Math.—Acta* 5(1964), 330—339).

10. __________. On the computation of the Nielsen number, *Pacific J. Math.*

11. __________. Lectures on Nielsen Fixed Point Theory, *Amer. Math. Society, Vol.* ***14*** (1983).

12. S.K. Kim, Dennis McGavran, Jingyal Pak. The Nielsen Numbers and Fiberings, Studies in Topology, Academic Press, Inc. 1975.

13. S. Lefschetz. On the Fixed Point Formula, *Ann. Math.* ***38***(1937), 819—822.

14. W.S. Massey. Algebraic Topology: An Introduction, Springer-Verlag, 1977.

15. Stasheff. On Homotopy Abelian H-spaces, *Proc. Camb. Phil. Soc.* ***57***(1961).

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