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Einstein CR-submanifolds of a flat Kaehlerian manifold

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Since A. Bejancu [1] introduced the notion of CR-submanifold of a Kaehlerian manifold, many authors have studied the CR-submanifold of a complex space form ([2], [3], [5], [6] ect.). The CR-submanifolds of a Kaehlerian manifold are the general notion of holomorphic, totally real and generic submanifolds (see §1).

The purpose of the present paper is to characterize Einstein CR-submanifold of an even-dimensional Euclidean space. Our main result appears in Theorem 2.2.

§1. CR-submanifolds of an even-dimensional Euclidean space

Let E be a 2m-dimensional Euclidean space and X the position vector representing a point P in E with respect to the origin 0. E being even-dimensional, it can be regarded as a flat Kaehlerian manifold, and hence there exists a tensor field F of type (1,1) and a Riemannion connection $\tilde{\nabla}$ on E satisfying

(1.1) $$F^2=-1,\ (FY)\cdot(FZ)=Y\cdot Z,\ \tilde{\nabla}F=0$$

for all vector fields Y and Z, where I denotes the identity transformation, and the dot the inner product in the Euclidean space E.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U;\ x^a\}$, where here and in the sequel the indices $a,\ b,\ c,\cdots$ run over the range $\{1,\ 2,\ \cdots,\ n\}$.

Furthermore, we assume that M is isometrically immersed in E by the immersion $i:M\longrightarrow E$ and identify $i(M)$ with M itself. Put $X_c=\partial_c X$, $\partial_c=\partial/\partial x^c$. Then X_c are n linearly independent local vector fields tangent to the submanifold M and thus $g_{cb}=X_c\cdot X_b$ are local components of the tensor representing the Riemannian metric induced on M from that of E.

Throughout the present paper, we denote by C_x $2m$-n mutually orthogonal unit normals to M. The indices u, v, w, x, y and z run over the range $\{n+1, n+2, \cdots, 2m\}$.

Denoting by $\{_c{}^a{}_b\}$ the Christoffel symbols formed with g_{cb} and by ∇_c the operator of the van der Waerden-Bortolotti covariant differentiation with respect to $\{_c{}^a{}_b\}$, equations of Gauss and and Weingarten for M is given respectively by

(1.2) $$\nabla_c X_b = h_{cb}{}^x C_x, \quad \nabla_c C_x = -h_c{}^a{}_x X_a$$

Since the ambient manifold E is Euclidean, the equation of Gauss, Codazzi and Ricci for M are respectively given by

(1.3) $$K_{dcb}{}^a = h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$$

(1.4) $$\nabla_c h_{ba}{}^x - \nabla_b h_{ca}{}^x = 0,$$

(1.5) $$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y,$$

The transformation of X_c and C_x by the almost complex structure tensor F are represented in each coordinate neighborhood as follows:

(1.6) $$FX_c = f_c{}^a X_a - f_c{}^x C_x,$$

(1.7) $$FC_x = f_x{}^a X_a + f_x{}^y C_y,$$

Applying F to (1.6) and (1.7) respectively and taking account of (1.1) and these equations, we find

(1.8) $$f_c{}^e f_e{}^a = -\delta_c^a + f_c{}^x f_x{}^a,$$

(1.9) $$f_c{}^e f_e{}^x + f_c{}^y f_y{}^x = 0, \quad f_x{}^e f_e{}^a + f_x{}^y f_y{}^a = 0,$$

(1.10) $$f_x{}^y f_y{}^z = -\delta_x^z + f_x{}^e f_e{}^z,$$

(1.11) $$g_{ed} f_c{}^d f_b{}^e = g_{cb} - f_c{}^x f_{bx}.$$

In a CR-submanifold M of E, the following theorem are well known ([6]).

Theorem 1.1. A necessary and sufficient condition for a submanifold M of a flat Kachlerian manifold to be a CR-submanifold is that the components of the tensor f and $f_c{}^x$ satisfy

(1.12) $$f_c{}^e f_e{}^b f_b{}^a + f_c{}^a = 0, \quad f_e{}^x f_c{}^e = 0.$$

And consequently we have from (1.9), (1.10) and (1.12) that

(1.13) $$f_c{}^x f_x{}^y = 0, \quad f_x{}^y f_y{}^z f_z{}^w + f_x{}^w = 0,$$

which shows that $f_c{}^a$ and $f_x{}^y$ determine an f-structure in M and that in the normal bundle of M respectively.

Differentiating (1.6) and (1.7) covariantly along M and making use of $\tilde{\nabla} F = 0$, (1.2) and these equations, we obtain

(1.14) $$\nabla_c f_b{}^a = h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x$$

(1.15) $$\nabla_c f_b{}^x = h_{ce}{}^x f_b{}^e - h_{cb}{}^z f_z{}^x,$$

(1.16) $\nabla_c f_x^a = h_c{}^a{}_y f_x{}^y - h_c{}^e{}_x f_e{}^a$

(1.17) $\nabla_c f_x{}^y = h_{cex} f^{ey} - h_{ce}{}^y f_x{}^e$

The aggregate ($f_c{}^a$, g_{cb}, $f_c{}^x$, $f_x{}^y$) satisfying (1.8)～(1.11) is said to be normal ([5]) if the second fundamental tensor $h_{cb}{}^x$ and the f-structure induced on M are commutative each other, that is,

(1.18) $h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e = 0.$

From now on we suppose that the normal connection of M is flat. Then we have $K_{dcy}{}^x = 0$, which together with (1.5) give

(1.19) $h_{ce}{}^x h_b{}^e{}_y - h_{be}{}^x h_c{}^e{}_y = 0.$

§2. Normal CR-submanifolds whose f-structure in the normal bundle is parallel

Let M be a normal CR-submanifold of an even-dimensional Euclidean space with flat normal connection, then by transvecting (1.18) with $f_d{}^b$ and remembering (1.8), we find $h_{cd}{}^x - (h_{ce}{}^x f_y{}^e) f_d{}^y - h_{be}{}^x f_c{}^e f_d{}^b = 0$, and then taking the skew-symmetric part with respect to indices c and d, we also have $(h_{ce}{}^x f_y{}^e) f_d{}^y - (h_{de}{}^x f_y{}^e) f_c{}^y = 0.$

If we transvect this with $f_z{}^d$ and take account of (1.10) and (1.13), then we obtain

(2.1) $h_{ce}{}^x f_z{}^e = P_{zy}{}^x f_c{}^y,$

where we have put

(2.2) $P_{zy}{}^x = h_{cb}{}^x f_z{}^c f_y{}^b.$

Using (2.1), the expression (1.17) reduces to $\nabla_c f_x{}^y = (P_w{}^y{}_x - P_{wx}{}^y) f_c{}^w.$

Throughout the present paper, we assume that the f-structure in the normal bundle is parallel, that is, $\nabla_c f_x{}^y = 0$. Then we have $(P_w{}^y{}_x - P_{wx}{}^y) f_c{}^w = 0$. Transvecting $f_z{}^c$ yields $P_z{}^y{}_x - P_{zx}{}^y = 0$, where we have used (1.10), (1.13) and (2.2). Hence, by putting $P_{zwy} = P_{zw}{}^x g_{xy}$ we see that P_{zwy} is symmetric for all indices. Therefore, it follows that

(2.3) $P_{zy}{}^x f_x{}^w = 0$

because of (1.13).

The normal connection being flat, by transvecting (1.19) with $f_z{}^b$ and considering (2.1), we obtain $(P_{uzy} P_w{}^{ux} - P^u{}_z{}^x P_{wuy}) f_c{}^w = 0$.

which together with (1.10) and (2.4) give

(2.4) $P_{uzy} P_w{}^{ux} - P_z{}^{ux} P_{wuy} = 0.$

Therefore we have

(2.5) $P_{uzy} P^{yuz} - P^u P_u = 0$

because P_{uzy} is symmetric for all indices, where we have put $P_u=g^{xy}P_{uxy}$.

Differentiating (2.1) covariantly along M and substituting (1.15)and (1.16) we find

$$(\nabla_d h_{ce}{}^x)f_z{}^e+h_{ce}{}^x(h_d{}^e{}_y f_z{}^y-h_d{}^a{}_z f_a{}^e)=(\nabla_d P_{zy}{}^x)f_c{}^y+P_{zy}{}^x(h_{de}{}^y f_c{}^e-h_{dc}{}^w f_w{}^y).$$

If we take the skew-symmetric part of this with respect to indices d and c, and use (1.4) and (1.19), then we get

$$-2h_{ce}{}^x h_d{}^a{}_z f_a{}^c=(\nabla_d P_{zy}{}^x)f_c{}^y-(\nabla_c P_{zy}{}^x)f_d{}^y+P_{zy}{}^x(h_{de}{}^y f_c{}^e-h_{ce}{}^y f_d{}^e),$$

or, take account of (1.18)

(2.6) $$2h_{ae}{}^x h_d{}^a{}_z f_c{}^e=(\nabla_d P_{zy}{}^x)f_c{}^y-(\nabla_c P_{zy}{}^x)f_d{}^y+2P_{zy}{}^x h_{de}{}^y f_c{}^e.$$

Transforming this with $f_w{}^c$ and remembering (1.10) and (1.12), we have

(2.7) $$(\nabla_d P_{yz}{}^x)(\delta_w{}^y+f_w{}^u f_u{}^y)=(f_w{}^e\nabla_e P_{zy}{}^x)f_d{}^y.$$

Since $f_y{}^x$ is parallel in the normal bundle, we see from (2.3) that

(2.8) $$(\triangle_d P_{yz}{}^x)f_x{}^w=0.$$

Thus (2.7) becomes

(2.9) $$\nabla_d P_{wz}{}^x=(f_z{}^e\nabla_e P_{yw}{}^x)f_d{}^y$$

because of $P_{yz}{}^x=P_{zy}{}^x$ and hence $(\nabla_d P_{wz}{}^x)f_c{}^w=(f_z{}^e\nabla_e P_{yw}{}^x)f_d{}^y f_c{}^w$.

Substituting this into (2.6), we get $h_{ae}{}^x h_d{}^a{}_z f_c{}^e=P_{zy}{}^x h_{de}{}^y f_c{}^e$.

Transforming this by $f_b{}^c$ and considering (1.8), (2.1) and (2.4), we find

(2.10) $$h_{ba}{}^x h_d{}^a{}_z=P_{zy}{}^x h_{bd}{}^y.$$

Transforming (2.9) by $f_d{}^c$ and remembering (1.12), we find

(2.11) $$f_c{}^e\nabla_c P_{yz}{}^x=0.$$

Transvecting (2.10) with $f_w{}^z f^{xw}$ and using (2.3), we get $\|h_{cb}{}^x f_x{}^w\|^2=0$ and thus

(2.12) $$h_{cb}{}^x f_x{}^w=0.$$

If we differentiate (2.9) covariantly and take account of (1.15), (2.11) and (2.12),

$$\nabla_c\nabla_d P_{yz}{}^x=(f_z{}^e\nabla_e P_{yw}{}^x)h_{ca}{}^y f_d{}^a+f_z{}^e(\nabla_c\nabla_e P_{yw}{}^x)f_d{}^y.$$

Since the normal connection is flat, we have

(2.13) $$2(f_z{}^e\nabla_e P_{yw}{}^x)h_{da}{}^y f_c{}^a-f_z{}^e(\nabla_c\nabla_e P_{yw}{}^x)f_d{}^y+f_z{}^e(\nabla_d\nabla_e P_{yw}{}^x)f_c{}^y=0$$

because of (1.18).

Transvecting (2.13) with $f_u{}^c$ and using (1.10), (1.12) and (2.8), we get

$$f_z{}^e\nabla_d\nabla_e P_{uw}{}^x=(f_u{}^c f_z{}^e\nabla_c\nabla_e P_{yw}{}^x)f_d{}^y.$$

Thus, it follows that $(f_z{}^e\nabla_d\nabla_e P_{uw}{}^x)f_c{}^w-(f_z{}^e\nabla_c\nabla_e P_{uw}{}^x)f_d{}^w=0$.

Therefore (2.13) turns out to be $(f_z{}^e\nabla_e P_{yw}{}^x)h_{da}{}^y f_c{}^a=0$.

Transvecting the above equation with $f_b{}^z$ and remembering (2.11). we obtain $(\nabla_b P_{yw}{}^x)h_{de}{}^y f_c{}^e=0$.

If we transvect this with $f_a{}^c$ and using (1.8) and (2.1), then we have

(2.14) $\nabla_c P_{yz}{}^x (h_{db}{}^z - P_{uv}{}^z f_d{}^u f_b{}^v) = 0.$

Lemma 2.1. Let M be a normal CR-submanifold of an even-dimensional Euclidean space with flat normal connection. If M is proper Einstein and the f-structure in the normal bundle is parallel, then the mean curvature vector of M is parallel.

Proof. Transvecting (2.14) with $h_w{}^{db}$ and using (2.2), (2.10) and (2.4), we get

(2.15) $\nabla_c P_{yz}{}^x (h_e{}^{ev} - P^v) P_{wv}{}^z = 0.$

On the other hand, we see from (1.3) that the Ricci tensor K_{cb} is of the form $K_{cb} = h_e{}^{ex} h_{cbx} - h_c{}^e{}_x h_{be}{}^x$, which together with (2.10) yields $K_{cb} = (h_e{}^{ex} - P^x) h_{cbx}$.

Transvecting this with $f_u{}^c f_v{}^b$ and considering (2.2), we have

$$K_{cb} f_u{}^c f_v{}^b = (h_e{}^{ex} - P^x) P_{uvx}.$$

From this and (2.15), we have $K_{cb} f_u{}^c f_v{}^b (\nabla_d P_{yz}{}^u) = 0$.

Since M is proper Einstein, that is, $K_{cb} = \frac{K}{n} g_{cb}$, where the scalar curvature K is not zero, it follows that $f_{ub} f_v{}^b (\nabla_d P_{yz}{}^u) = 0$,

which implies that $\nabla_d P_{yz}{}^x = 0$ because of (2.8). Differentiating (2.2) covariantly and taking account of the last equation, we get

$$(\nabla_d h_{cb}{}^x) f_y{}^c f_z{}^b + h_b{}^{cx} (\nabla_d f_{cy}) f_z{}^b + h_b{}^{cx} (\nabla_d f_{bz}) f_y{}^c = 0,$$

which together with (1.15), (2.1) and (2.3) give

(2.16) $(\nabla_d h_{cb}{}^x) f_y{}^c f_z{}^b = 0.$

Differentiating (1.18) covariantly and using (1.14), (2.2) and (2.10), we find

$$(\nabla_d h_{ce}{}^x) f_b{}^e + (\nabla_d h_{be}{}^x) f_c{}^e = 0.$$

If we take the skew-symmetric part of this with respect to indices d and c, and using the equation of Codazzi, then we obtain $(\nabla_d h_{be}{}^x) f_c{}^e - (\nabla_c h_{be}{}^x) f_d{}^e = 0$.

The last two equations implies $(\nabla_d h_{be}{}^x) f_c{}^e = 0$

because of (1.4). Transforming this by f^{cb} and remembering (1.8), we get

(2.17) $\nabla_d h_a{}^{ax} - (\nabla_d h_{be}{}^x) f^{ez} f_z{}^b = 0.$

Therefore, we see from (2.16) and (2.17) that the mean curvature vector is parallel in the normal bundle. Hence Lemma 2.1 is proved.

Combining Theorem 4.1 of [5] and Lemma 2.1, the complete submanifold M is of the form:

$$S^n(r) \text{ or } S^{p_1}(r) \times \cdots \times S^{p_N}(r), \quad p_1 + \cdots + p_N = n, \quad 1 < N \leqq 2m - n$$

because the scalar curvature of M is not zero. But, M being Einstein, we see that $p_1 = \cdots = p_N$, $nN = n$, p is an odd number $\geqq 1$.

Thus we have

Theorem 2.2. Let M be an n-dimensional complete proper Einstein and normal CR-submanifold of an even-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure in the normal bundle is parallel, then M is a sphere $S^n(r)$ of dimension n, or a pythagorean product of the form;

$$S^n(r)\times\cdots\times S^p(r),\ p \text{ is an odd number} \geqq 1,\ N_p=n,$$

$1<N\leqq 2m-n$, where $S^p(r)$ is a p-dimensional sphere with radius $r>0$. If M is a pythagorean product of the above form, then M is of essential codimension N.

Corollary 2.3([4]). Let M be an n-dimensional complete proper Einstein and generic submanifold of an even-dimensional Euclidean space with flat normal connection. Then M is the same type as that of Theorem 2.2.

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