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GENERIC SUBMANIFOLDS OF A QUATERNIONIC PROJECTIVE SPACE WITH COMMUTATIVE SECOND FUNDAMENTAL FORMS

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A submanifold M of a quaternionic projective space QP^m of real dimension $4m$ is called a *generic submanifold* if the normal space $N_P(M)$ of M at P is always mapped into the tangent space $T_P(M)$ at P under the action of the quaternionic Kaehlerian structure tensors of the ambient manifold at the same time (Cf. Y. Shibuya [8]). Any real hypersuface of a quaternionic projective space is a generic submanifold.

In 1970, Lawson [4] determined real hypersurfaces of QP^m by using the method of Riemannian fibre bundles which are compatible with the Hopf-fibration $S^{4m+3}(1)\to QP^m$, where S^{4m+3} (1) denotes the unit sphere of dimension $4m+3$. Recently Shibuya [8] developed the method of Lawson in the case of generic submanifolds immersed in QP^m. By using the same method the present author [6], [7] and Kang [7] have studied the following theorems:

THEOREM A. (see [7]). *Let M be an n-dimensional complete, generic submanifold of a quaternionic projective space $QP^{(n+p)/4}$ with flat normal connection. If (2.1) appeared in §2 are valid at each point of M and if the mean curvature vector defined of M is parallel in the normal bundle, then M is of the form*

$$\pi(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),$$

$p_1, \cdots, p_N\leqq 1$, $p_i=4l_i+3$ *(l_i : non-negative integer)*, $\sum_i p_i=n+3$, $\sum_i r_i^2=1$, $N=p+1$, *where π denote the natural projection of $S^{4m+3}(1)$ onto QP^m which is defined by the Hopf-fibration.*

THEOREM B. (see [5]). *Let M be a complete real hypersurface of QP^m. If (2.1)) appeared in §2 are valid at each point of M, then M is of the form*

$$\pi(S^{4p+3}(a)\times S^{4q+3}(b))$$

for some portion (p, q) of $m-1$ and some a, b such that $a^2+b^2=1$.

Dedicated to Professor Sang-Seup Eum on his 60th birthday

For a generic submanifold M of a quaternionic projective space, it is easily seen from the equation (1.13) of Ricci that there is no relationship between the flatness of the normal connection of M and the commutativity of the second fundamental forms.

In the present paper we study generic submanifolds of quaternionic projective spaces with commutative second fundamental forms and determine such submanifolds by using Theorem B (see Theorem 4).

Manifolds, submanifolds, geometric objects and mappings we dicuss in this paper will be assumed to be differentiable and of class C^{∞}.

1. Generic submanifolds of a quaternionic Kaehlerian manifold

Let $\overline{M}$ be an $(n+p)$-dimensional quaternionic Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\overline{U};\ y^i\}$ (here and in the sequel the indices h, i, j, k, t, s run over the range $\{1, 2, \cdots, n+p\}$. Then, by definition there exists a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over $\overline{M}$ satisfying the following conditions (1), (2) and (3):

(1) In any coordinate neighborhood $\overline{U}$, there is a local base $\{F, G, H\}$ of V such that

$$F_h^iF_j^h=-\delta_j^i,\ G_h^iG_j^h=-\delta_j^i,\ H_h^iH_j^h=-\delta_j^i, \tag{1.1}$$

$$G_h^iH_j^h=-H_h^iG_j^h=F_j^i,\ F_h^iG_j^h=-G_h^iF_j^h=H_j^i,\ H_h^iF_j^h=-F_i^hH_j^h=G_j^i,$$

where F_j^i, G_j^i and H_j^i denote the local components of F, G and H in $\overline{U}$ respectively.

(2) There is a Riemannian metric tensor g_{ji} such that

$$F_{ji}=-F_{ij},\ G_{ji}=-G_{ij},\ H_{ji}=-H_{ij}, \tag{1.2}$$

where $F_{ji}=g_{ih}F_j^h$, $G_{ji}=g_{ih}G_j^h$ and $H_{ji}=g_{ih}H_j^h$.

(3) For the operator ∇_j of the covariant differentiation with respect to the Riemannian connection

$$\begin{aligned}\nabla_jF_i^h&=r_jG_i^h-q_jH_i^h,\\ \nabla_jG_i^h&=-r_jF_i^h+p_jH_i^h,\\ \nabla_jH_i^h&=q_jF_i^h-p_jG_i^h,\end{aligned} \tag{1.3}$$

where $p=p_idy^i$, $q=q_idy^i$ and $r=r_idy^i$ are certain local 1-forms defined in $\overline{U}$.

Such a local base $\{F, G, H\}$ is called a ***canonical base*** of the bundle V in $\bar{U}$ (see Ishihara [1] etc.).

Let M be an n-dimensional submanifold immersed in $\bar{M}$. Then M is covered by a system of coordinate neighborhoods $\{U; x^a\}$, $U=\bar{U}\cap M$ (here and in the sequel the indices $a, b, c, d, e \cdots$ run over the range $\{1, 2, \cdots, n\}$). Let M be represented by $y^i=y^i(x^a)$ with respect to local coordinates (y^i) in $\bar{U}$ and (x^a) in U. We put

$$B_a^i=\partial_a y^i \quad (\partial_a=\partial/\partial x^a)$$

and denote by N_x^i p mutually orthogonal unit vectors normal to M (here and in the sequel the indices $x, y, z, v, w \cdots$ run over the range $\{n+1, \cdots, n+p\}$). Putting

$$g_{ba}=g_{ji}B_b^jB_a^i, \quad g_{yx}=g_{ji}N_y^jN_x^i,$$

they are the induced metric tensors of M and of the normal bundle respectively.

If the transforms by F, G and H of any vector normal to M are always tangent to M at the same time, then the submanifold M is called a ***generic submanifold*** (see Shibuya [8] etc.). In such case, since the ranks of F, G and H are all $u+p$, we have $n\geqq 3p$.

For a generic submanifold M of $\bar{M}$, we can put in each coordinate neighborhood U

$$\begin{aligned} &F_h^iB_a^h=\phi_a^bB_b^i+\phi_a^xN_x^i, \quad F_h^iN_x^h=-\phi_x^aB_a^i, \\ (1.4)\qquad &G_h^iB_a^h=\Psi_a^bB_b^i+\Psi_a^xN_x^i, \quad G_h^iN_x^h=-\Psi_x^aB_a^i, \\ &H_h^iB_a^h=\theta_a^bB_b^i+\theta_a^xN_x^i, \quad H_h^iN_x^h=-\theta_x^aB_a^i. \end{aligned}$$

As a consequence of (1.2), we have from (1.4)

$$\begin{aligned} (1.5)\qquad &\phi_{ba}=-\phi_{ab}, \quad \Psi_{ba}=-\Psi_{ab}, \quad \theta_{ba}=-\theta_{ab} \\ &\phi_{ax}=\phi_{xa}, \quad \Psi_{ax}=\Psi_{xa}, \quad \theta_{ax}=\theta_{xa}, \end{aligned}$$

where we have put $\phi_{ba}=\phi_b^eg_{ea}$, $\phi_{ax}=\phi_a^yg_{yx}$ etc.

Applying F, G, H to (1.4) and using (1.1) and those equations, it follows that

$$\begin{aligned} &\phi_b^e\phi_a^e=-\delta_a^b+\phi_a^x\phi_x^b, \quad \Psi_e^b\Psi_a^e=-\delta_a^b+\Psi_a^x\Psi_x^b, \quad \theta_e^b\theta_a^e=-\delta_a^b+\delta_a^b+\theta_a^x\theta_x^b, \\ &\phi_a^e\phi_e^x=0, \qquad \Psi_a^e\Psi_e^x=0, \qquad \theta_a^e\theta_e^x=0. \end{aligned}$$

$$
\begin{aligned}
&\Psi^b_e\phi^e_a=-\theta^b_a+\phi^x_a\Psi^b_x,\ \ \theta^b_e\Psi^e_a=-\phi^b_a+\Psi^x_a\theta^b_x,\ \ \phi^b_e\theta^e_a=-\Psi^b_a+\theta^x_a\phi^b_x,\\
(1.6)\quad &\phi^b_e\Psi^e_a=\theta^b_a+\Psi^x_a\phi^b_x,\ \ \Psi^b_e\theta^e_a=\phi^b_a+\theta^x_a\Psi^b_x,\ \ \theta^b_e\phi^e_a=\Psi^b_a+\phi^x_a\theta^b_x,\\
&\theta^e_a\phi^x_e=-\Psi^x_a,\ \ \Psi^e_v\theta^x_e=-\phi^x_a,\ \ \phi^e_a\Psi^x_e=-\theta^x_a,\\
&\phi^e_a\theta^x_e=\Psi^x_a,\ \ \theta^e_a\Psi^x_e=\phi^x_a,\ \ \Psi^e_a\phi^x_e=\theta^x_a,\\
&\phi^a_x\Psi^y_a=0,\ \ \Psi^a_x\theta^y_a=0,\ \ \theta^a_x\phi^y_a=0,\ \ \phi^a_x\phi^y_a=\delta^y_x,\ \ \Psi^a_x\Psi^y_a=\delta^y_x,\ \ \theta^a_x\theta^y_a=\delta^y_x.
\end{aligned}
$$

We denote by ∇_b the operator of the covariant differentiation with respect to the Riemannian connection induced on M from the connection of $\overline{M}$. Then the equations of Gauss and Weingarten are given by

$$
(1.7)\qquad \nabla_b B^i_a=h_{ba}{}^x N^i_x,\ \ \nabla_b N^i_x=-h_{ba}{}^x B^i_a
$$

respectively, where $h_{ba}{}^x$ are components of the second fundamental tensor with respect to the unit normal vector N^i_x and $h_{ba}{}^x=h_{be}{}^y g^{ea} g_{yx}$.

Applying the operator $\nabla_c=B^i_c\nabla_i$ to (1.4) and using (1.3) and (1.7), we can see that

$$
\begin{aligned}
&\nabla_c\phi^b_a=r_c\Psi^b_a-q_c\theta^b_a+h^x_{cb}\phi^x_a-h^x_{ac}\phi^b_x,\\
(1.8)\quad &\nabla_c\Psi^b_a=-r_c\phi^b_a+p_c\theta^b_a+h^x_{cb}\Psi^x_a-h^x_{ca}\Psi^b_x,\\
&\nabla_c\theta^b_a=q_c\phi^b_a-p_c\Psi^b_a+h^x_{cb}\theta^x_a-h^x_{ca}\theta^b_x,\\
&\nabla_c\phi^x_a=r_c\Psi^x_a-q_c\theta^x_a-h^x_{ce}\phi^e_a,\\
(1.9)\quad &\nabla_c\Psi^x_a=-r_c\phi^x_a+p_c\theta^x_a-h^x_{ce}\Psi^e_a,\\
&\nabla_c\theta^a_x=q_c\phi^x_a-p_c\Psi^x_a-h^x_{ce}\theta^e_a\\
(1.10)\quad &h^x_{ce}\phi^y_e=h^y_{ce}\phi^e_x,\ \ h^x_{ce}\Psi^y_e=h^y_{ce}\Psi^e_x,\ \ h^x_{ce}\theta^y_e=h^y_{ce}\theta^e_x,
\end{aligned}
$$

where $p_c=p_iB^i_c$, $q_c=q_iB^i_c$ and $r_c=r_iB^i_c$.

If the ambient manifold $\overline{M}$ is a quaternionic Kaehlerian manifold with constant Q-sectional curvature c, then $K_{kji}{}^h$ the components of the curvature tensor of $\overline{M}$ are of the form

$$
\begin{aligned}
K_{kji}{}^h=\frac{c}{4}(&\delta^h_k g_{ji}-\delta^h_j g_{ik}+F^h_k F_{ij}-F^h_j F_{ki}-2F_{kj}F^h_i+G^h_k G_{ji}\\
&-G^h_j G_{ki}-2G_{kj}G^h_i+H^h_k H_{ji}-H^h_j H_{ki}-2H_{kj}H^h_i),
\end{aligned}
$$

where c is necessarily a constant, provided $n+p\geqq 8$ (see Ishihara [1]). Thus the structure equations of Gauss, Codazzi and Ricci are given respectively by

$$(1.11)\qquad K_{dcb}{}^{a}=\frac{c}{4}(\delta_d^a g_{cb}-\delta_c^a g_{db}+\phi_d^a\phi_{cb}-\phi_c^a\phi_{bd}-2\phi_{dc}\phi_b^a+\Psi_d^a\Psi_{cb}-\Psi_d^a\Psi_{cb}$$
$$-2\Psi_{dc}\Psi_b^a+\theta_d^a\theta_{cb}-\theta_c^a\theta_{db}-2\theta_{dc}\theta_b^a)+h_{da}{}^{x}h_{cb}{}^{x}-h_{ca}{}^{x}h_{db}{}^{x},$$

$$(1.12)\qquad \nabla_c h_{ba}{}^{x}-\nabla_b h_{ca}{}^{x}=\frac{c}{4}(\phi_c^x\phi_{ba}-\phi_b^x\phi_{ca}-2\phi_{cb}\phi_a^x+\Psi_c^x\Psi_{ba}-\Psi_b^x\Psi_{ca}$$
$$-2\Psi_{cb}\Psi_a^x+\theta_c^x\theta_{ba}-\theta_b^x\theta_{ca}-2\theta_{cb}\theta_a^x),$$

$$(1.13)\qquad K_{cby}{}^{x}=\frac{c}{4}(\phi_c^x\phi_{by}-\phi_b^x\phi_{cy}+\Psi_c^x\Psi_{by}-\Psi_b^x\Psi_{cy}+\theta_c^x\theta_{by}-\theta_b^x\theta_{cy})$$
$$+h_{ce}{}^{x}h_{by}{}^{e}-h_{be}{}^{x}h_{cy}{}^{e},$$

where $K_{dcb}{}^{a}$ and $K_{cby}{}^{x}$ denote components of the curvature tensors determined by the induced metric g_{ba} and g_{yx} in M and in the normal bundle of M respectively.

2. Some properties concerning with commutativity of the second fundamental forms

Let M be an n-dimensional generic submanifold of an $(n+p)$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c $(n+p\geqq 8)$ which satisfies

$$(2.1)\qquad h_{be}{}^{x}\phi_a^e+h_{ae}{}^{x}\phi_b^e=0,\ \ h_{be}{}^{x}\Psi_a^e+h_{ae}{}^{x}\Psi_b^e=0,\ \ h_{be}{}^{x}\theta_a^e+h_{ae}{}^{x}\theta_b^e=0$$

at every point of M. For a real hypersurface, denoting by $h_{ba}=h_{ba}^1$, the condition (2.1) reduces to

$$(2.1)\qquad h_{be}\phi_a^e+h_{ae}\phi_b^e=0,\ \ h_{be}\Psi_a^e+h_{ea}\Psi_b^e=0,\ \ h_{be}\theta_a^e+h_{ae}\theta_b^e=0.$$

From now on we assume that M has commutative second fundamental forms, that is,

$$(2.2)\qquad h_{ce}{}^{x}h_{b}{}^{e}{}_{y}=h_{be}{}^{x}h_{c}{}^{e}{}_{y}$$

are valid at every point of M.

We transvect the first equation of (2.1) with ϕ_y^a. Then, making use of (1.6), we have

$$\phi_y^a h_{ae}{}^{x}\phi_c^e=0,$$

from which, transvecting with ϕ_c^b, (2.3) $$h_{be}{}^{x}\phi^{e}_{y}=P_{yz}{}^{x}\phi^{z}_{b},$$

where here and in the sequel we put

(2.4) $$P_{yz}{}^{x}=h_{ab}{}^{x}\phi^{b}_{y}\phi^{a}_{z}.$$

From the other equations of (2.1) we can similarily obtain

(2.5) $$h_{be}{}^{x}\Psi^{e}_{y}=Q_{yz}{}^{x}\Psi^{z}_{b},\ \ h_{be}{}^{x}\theta^{e}_{y}=R_{yz}{}^{x}\theta^{z}_{b},$$

where $Q_{yz}{}^{x}=h_{ba}{}^{x}\Psi^{b}_{y}\Psi^{a}_{z}$ and $R_{yz}{}^{x}=h_{ba}{}^{x}\theta^{b}_{y}\theta^{a}_{z}$. By the way, transvecting the second equation of (2.1) with ϕ^{b}_{y}, we get

$$P_{yz}{}^{x}\phi^{z}_{e}\Psi^{e}_{a}-h_{ae}{}^{x}\theta^{e}_{y}=0,$$

which and (2.5) yield

$$(P_{yz}{}^{x}-R_{yz}{}^{x})\theta^{z}_{a}=0$$

and consequently

$$P_{yz}{}^{x}=R_{yz}{}^{x}.$$

Hence, by the quite similar way, we can see that

(2.6) $$P_{yz}{}^{x}=Q_{yz}{}^{x}=R_{yz}{}^{x}.$$

We now apply the operator ∇_c to (2.3) and take the skew-symmetric part of the equation thus obtained. Then it follows that

$$(\nabla_c h_{ba}{}^{x}-\nabla_b h_{ca}{}^{x})^{a}_{y}+h_{ba}{}^{x}\nabla_c\phi^{a}_{y}-h_{ca}{}^{x}\nabla_b\phi^{a}_{y}=(\nabla_c P_{yz}{}^{x})\phi^{z}_{b}-(\nabla_b P_{yz}{}^{x})\phi^{z}_{c}+P_{yz}{}^{x}(\nabla_c\phi^{z}_{b}-\nabla_b\phi^{z}_{c}),$$

from which, substituting (1.9) and (1.12), we have

(2.7) $$\frac{c}{4}(-2\phi_{cb}\delta^{x}_{y}+\Psi^{x}_{c}\theta_{by}-\Psi^{x}_{b}\theta_{cy}-\theta^{x}_{c}\Psi_{by}+\theta^{x}_{b}\Psi_{cy})+h_{ba}{}^{x}h_{ce}{}^{y}\phi^{a}_{e}-h_{ca}{}^{x}h_{be}{}^{y}\phi^{a}_{e}$$
$$=(\nabla_c P_{yz}{}^{x})\phi^{z}_{b}-(\nabla_b P_{yz}{}^{x})\phi^{z}_{c}+P_{yz}{}^{x}(-h_{ce}{}^{z}\phi^{e}+h_{be}{}^{z}\phi^{e}_{c}).$$

Transvecting (2.7) with ϕ^{b}_{w} and using (1.6) and (2.1), we have

$$\nabla_c P_{yw}{}^{x}=\phi^{b}_{w}(\nabla_b P_{yz}{}^{x})\phi^{z}_{c},$$

from which, transvecting with ϕ^{y}_{a},

(2.8) $$(\nabla_c P_{yw}{}^{x})\phi^{y}_{a}=\phi^{b}_{w}(\nabla_b P_{yz}{}^{x})\phi^{z}_{c}\phi^{y}_{a}.$$

On the other hand, putting $P_{yzx}=P_{yz}{}^{w}g_{wx}$, it easily follows from (1.10) and the definition (2.4) of $P_{yz}{}^{x}$ that P_{yzx} are symmetric with respect to all indices. Hence (2.8) gives

$$(\nabla_c P_{yw}{}^x)\phi_a^y=(\nabla_a P_{yw}{}^x)\phi_c^y,$$

which and (2.7) imply

$$\frac{c}{8}(-2\phi_{cb}\delta_y^x+\Psi_c^x\theta_{by}-\Psi_b^x\theta_{cy}-\theta_c^x\Psi_{by}+\theta_b^x\Psi_{cy})+h_{ae}^x h_{by}^e\phi_c^a=P_{yz}^x h_{be}^z\phi_c^e, \tag{2.9}$$

where we have used the assumption that the second fundamental tensors are commutative i.e. that (2.2) is valid at every point of M.

We transvect (2.9) with ϕ_d^c and use (1.6). Then we have

$$\begin{aligned}\frac{c}{8}\{2(g_{db}-\phi_d^z\phi_{zb})\delta_y^x-\Psi_d^x\Psi_{by}-\Psi_b^x\Psi_{dy}-\theta_b^x\theta_{dy}-\theta_d^x\theta_{dy}\}-h_{de}{}^x h_{by}{}^e \\ +h_{ae}{}^x h_b{}^e{}_y\phi_d^z\phi_z^a=-P_{yz}{}^x h_{db}{}^z+P_{yz}{}^x h_{be}{}^z\phi_d^w\phi_w^e.\end{aligned} \tag{2.10}$$

On the other hand, a direct computation by using (2.2) and (2.3) implies

$$\begin{aligned}P_{yz}{}^x h_{be}{}^z\phi_d^w\phi_w^e&=h_{ca}{}^x\phi_y^c\phi_z^a h_{ae}{}^z\phi_d^w\phi_w^e\\&=h_{ca}{}^x\phi_y^c\phi_z^a h_{bew}\phi_d^w\phi^{ze}\\&=h_{ca}{}^x\phi_y^c(g^{ae}+\phi_s^a\phi^{es})h_{bew}\phi_d^w\\&=h_{ca}{}^x\phi_y^c h_{bz}{}^a\phi_a^z\\&=h_{ba}{}^x h_{cz}{}^a\phi_y^c\phi_d^w\\&=h_{ac}{}^x h_{by}{}^a\phi_z^c\phi_d^z\end{aligned}$$

with the help of (1.10). Consequently (2.10) becomes

$$h_{be}{}^x h_{ay}^e=P_{yz}{}^x h_{ba}{}^z+\frac{c}{8}\{2(g_{ba}-\phi_b^a-\phi_{za})\delta_y^x-\Psi_b^x\Psi_a^y-\Psi_a^x\Psi_{by}-\theta_b^x\theta_{ay}-\theta_a^x\theta_{by}\} \tag{2.11}$$

and also

$$h_{ba}^x h^{ba}{}_x=P_x h^x+\frac{c}{4}p(n-p-2), \tag{2.12}$$

where here and in the sequel we put $h^x=h_{ba}{}^x g^{ba}$, $P_x=P_{yzx}g^{yz}$ and $h^{ba}{}_x=h_c{}^a{}_x g^{cb}$.

We next prove

LEMMA 1. *Let M be an n-dimensional generic submanifold of an $(n+p)$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature $c(n+p\geqq 8)$. If (2.1) are valid and the second fundamental tensors are commutative at each point of M, then*

$$\nabla_c P^x=\nabla_c h^x. \tag{2.13}$$

PROOF. Applying the operator ∇_c to the first equation of (2.1), we have

$$(\nabla_c h_{be}{}^x)\phi^e_a + h_{be}{}^x(h_c{}^e{}_y\phi^y_a - h_{ca}{}^y\phi^e_y) + (\nabla_c h_{ae}{}^x)\phi^e_b + h_{ae}{}^x(h_c{}^e{}_y\phi^y_b - h_{cb}{}^y\phi^e_y) = 0,$$

from which, substituting (2.11),

$$(\nabla_c h_{be}{}^x)\phi^e_a + (\nabla_c h_{ae}{}^x)\phi^e_b + \frac{c}{8}(\{2(g_{bc} - \phi^z_b\phi^c_z)\delta^x_y - \Psi^x_b\Psi_{cy} - \Psi^x_c\Psi_{by} - \theta^x_b\theta_{cy} - \theta^x_c\theta_{by}\}\phi^y_a$$
$$+ \{2(g_{ac} - \phi^z_a\phi_{zc})\delta^x_y - \Psi^x_b\Psi_{cy} - \Psi^x_c\Psi_{ay} - \theta^x_a\theta^y_c - \theta_{cx}\theta^y_a\}\phi^y_b) = 0.$$

Transvecting this equation with ϕ^a_d, we find

$$-\nabla_c h_{bd}{}^x + (\nabla_c h_{be}{}^x)\phi^y_d\phi^e_y + (\nabla_c h_{ae}{}^x)\phi_{da}\phi^e_b + \{2\phi_{dc}\delta^x_y + \theta^x_d\Psi_{cy} + \Psi^x_c\theta_{dy} - \Psi^x_d\theta_{cy} - \theta^x_c\Psi_{dy}\}\phi^y_b = 0,$$

from which, contracting with respect to the indices c and d and using the equation (1.12) of Codazzi, it can be easily verified that

$$\nabla_b h^x = (\nabla_b h_{ce}{}^x)\phi^{cy}\phi^e_y. \tag{2.14}$$

On the other hand

$$P^x = h_{ce}{}^x\phi^c_y\phi^{ye},$$

from which, applying the operator ∇_b and using (1.6) and (2.3), we can obtain

$$\nabla_b P^x = (\nabla_b h_{ce}{}^x)\phi^{cy}\phi^e_y = \nabla_b h^x.$$

3. Main theorems

Before we state our main results we introduce the following lemma, which is a direct consequence of the equation (1.12) of Codazzi.

LEMMA 2. (for details see [6]). *On an n-dimensional generic submanifold of a quaternionic Kaehlerian manifold of dimension $(n+p) \geqq 8$) with constant Q-sectional curvature c, the following inequalities are valid*:

$$\|\nabla_c h_{ba}{}^x\|^2 \geqq \frac{3}{8}c^2 p(n-p-2).$$

Let M be an n-dimensional generic submanifold of an $(n+p)$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c $(n+p \geqq 8)$. Suppose that at each point of M (2.1) are valid and the second fundamental tensors are commutative.

Now we compute the Laplacian ΔL of the function $L = h_{ab}{}^x h^{ba}{}_x$ globally defined on M, where $\Delta = g^{dc}\nabla_d\nabla_c$. Then we have by definition

$$\frac{1}{2}\Delta L = g^{dc}(\nabla_d\nabla_c h_{ba}{}^x)h^{ba}{}_x + \|\nabla_c h_{ba}{}^x\|^2,$$

or using (1.12) and Ricci identity,

$$\frac{1}{2}\Delta L=(\nabla_b\nabla_a h^x)h^{ba}{}_x+K_b^e h_{ea}{}^x h^{ba}{}_x-K_{cba}{}^e h_e{}^{cx}h^{ba}{}_x+\frac{c}{4}\{3(-h_e{}^{cx}\phi_b^e\phi_{ac}-h_{ce}{}^x\Psi_b^e\Psi^c{}_a-h_{ce}{}^x\theta_b^e\theta_a^c)$$
$$+3\phi_b^x(h_y\phi_a^y-h_{ca}{}^y\phi_y^c)+3\Psi_b^x(h_y\Psi^y{}_a-h_{ca}{}^y\Psi^c{}_y)+3\theta_b^x(h_y\theta_a^y-h_{ca}{}^y\theta_y^c)\}h^{ba}{}_x+\|\nabla_c h_{ba}{}^x\|^2.$$

with the help of (1.8) and (1.9), where K_b^a denotes the local component of the Ricci tensor of M which is of the form

$$K_b^a=\frac{c}{4}\{(n-8)\delta_b^a-3(\phi_b^y\phi_y^a+\Psi_b^y\Psi^a{}_y+\theta_b^y\theta_y^a)\}+h_x h_b{}^{ax}-h_b{}^e{}_x h_e{}^{ax}$$

because of (1.11).

Hence, by using (1.11), (2.1) and (2.2), we can easily obtain

$$\text{(3.1)}\qquad \frac{1}{2}\Delta L=(\nabla_b\nabla_a h^x)h^{ba}_x+\frac{c}{4}\{(n-9)h_{ba}{}^x h^{ba}_x-h^x h_x+3h_y(\phi_b^x\phi_a^y+\Psi_b^x\Psi^y_a$$
$$+\theta_b^x\theta_a^y)h^{ba}{}_x\}+h_y h_b^{ey}h_{ea}{}^x h^{ba}{}_x-(h_{ce}{}^y h^{ce}{}_x)(h_{bay}h_b{}^{ax})+\|\nabla_c h_{ba}{}^x\|^2.$$

On the other hand, a simple calculation by using (2.11) gives the following identities:

$$\text{(3.2)}\qquad h_y h_b^{ey}h_{ea}{}^x h^{ba}{}_x=h_y P_z P^{yz}{}_x h^x+\frac{c}{4}\{(n-2p-4)h_x P^x+ph_x h^x\},$$

$$\text{(3.3)}\qquad (h_{ce}{}^y h^{ce}{}_x)(h^{ba}{}_y h_{ba}{}^x)=P_{xz}{}^y P_{yw}{}^x h^z h^w+\frac{c}{2}(n-p-2)P_x h^x+p\{\frac{c}{4}(n-p-2)\}^2.$$

Substituting (3.2) and (3.3) into (3.1), it follows that

$$\text{(3.4)}\qquad \frac{1}{2}\Delta L=(\nabla_b\nabla_a r^x)h^{ba}{}_x+\frac{c}{4}(p-1)h_x h^x+\frac{c^2}{16}p(p-1)(n-p-2)+P_{yz}{}^x P_z h^x r^y$$
$$-P_{xz}{}^y P_{yw}{}^x h^z h^w+\{\|\nabla_c h_{ba}{}^x\|^2-\frac{3}{8}c^2 p(n-p-2)\}.$$

Moreover, by means of (2.1), (2.4) and (2.11), it is easily verified that

$$\begin{aligned}P_{xz}{}^y P_{yw}{}^x h^z h^w&=h_{bc}{}^y\phi_x^b\phi_z^c h_{ad}{}^x\phi_y^a\phi_w^d h^z h^w\\&=h_{bc}{}^z h_{ad}{}^x(\phi_e^c\phi^{ae}+g^{ca})\phi_x^b\phi_w^d h^z h^w\\&=h_{bcx}h_d^{cx}\phi_z^b\phi_w^d h^z h^w\\&=(P_x h_{bd}{}^x+\frac{c}{8}\{2p(g_{bd}-\phi_b^x\phi_{xd})-2\Psi_b^x\Psi^d{}_x-2\theta_b^x\theta_{xd})\times\phi_z^b\phi_w^d h^z h^w\\&=P_x P_{zw}{}^x h^z h^w.\end{aligned}$$

Consequently substituting this equation into (3.4) yields

$$\text{(3.5)}\qquad \frac{1}{2}\Delta L=(\nabla_b\nabla_a h^x)h^{ax}{}_b+\frac{c}{4}(p-1)h_x h^x+\left(\frac{c}{4}\right)^2 p(p-1)(n-p-2)$$

$$+\{\|\nabla_c h_{ba}^x\|^2-\frac{3}{8}c^2p(n-p-2)\}.$$

Thus we have

THEOREM 3. *Let M be an n-dimensional generic submanifold of an $(n+p)$-dimensional quaternionic Kaehlerian manifold $(n+p\geqq 8)$with constant Q-sectional curvature c $(\geqq 0)$. Suppose that the mean curvature vector of M is parallel in the normal bundle and the second fundamental tensors are commutative. If the equation* (2.1) *are valid at each point of M, then the ambient manifold is a Euclidean space or the submanifold M is a real hppersurface.*

PROOF. If the mean curvature vector $H^i=h^xN_x^i$ is parallel in the normal bundle, then $\nabla_b h^x=0$, and consequently the function L is constant because of (2.11) and Lemma 1. Thus, Lemma 2 and (3.5) imply

$$c(p-1)(n-p-2)=0$$

because $c\geqq 0$. Since $n\geqq 3p$, $u-p-2\geqq 2(p-1)$ and consequently

$$c(p-1)^2=0.$$

Hence we have $c=0$ or $p=1$.

Combining Theorem B and Theorem 3, we have

THEOREM 4. *Let M be an n-dimensional complete, generic submanifold of a quaternionic projective space $QP^{(n+p)/4}$. Suppose that the mean curvature vector of M is parallel in the normal bundle and that the second fundamental tensors are commutative. If the equation* (2.1) *are valid at each point of M, then the submanifold M is a real hypersurface and is of the form*

$$\pi(S^{4s+3}(r_1)\times S^{4t4+3}(r_2))$$

for some portion (s, t) of $m-1$ $(m=n+1)$ and some r_1, r_2 such that $r_1^2+r_2^2=1$.

REMARK. When the ambient manifold is a Euclidean space, such submanifolds are determined by Ki and the present author in their paper [3].

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