ON THE NUMERICAL RANGES AND
LUMER'S FORMULA

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1. Introduction

In [4], Kim and Yang defined a numerical range for the class of all numerically bounded (nonlinear) maps on a Hilbert $C^*$-module, and gave some of the basic properties of such numerical range. In this paper which is a continuation of [4], we define the numerical range for the numerically bounded vector fields on the unit sphere of a Hilbert $C^*$-module, and give additional properties of such numerical ranges. In particular we obtain an analogue of Lumer's formula for the class of Lipschitz maps.

Throughout this paper we let $B$ be a unital $C^*$-algebra, $B^*$ its dual space, and $X$ the Hilbert $B$-module with a $B$-valued inner product $\langle,\rangle$. A Hilbert $B$-module $X$ is assumed to have a vector space structure over the complex numbers $C$ compatible with that of $B$ in the sense that

$$\lambda(xb) = (\lambda x)b = x(\lambda b) \ (x \in X, \ b \in B, \ \lambda \in C).$$

We define the norm $\|\cdot\|_X$ on $X$ by $\|x\|_X = \|\langle x, x \rangle\|^{\frac{1}{2}}$. We will use the following notations. $Q^*(X)$ is the vector space of all $B^*$-quasibounded maps. $W^*(X)$ is the vector space of all $B^*$-numerically bounded maps. $L(X)$ is the Banach space of all bounded linear operators on $X$. We also denote the operator norm on $L(X)$ by $\|\cdot\|$. 

2. Numerical range for nonlinear operators

If we set

\[ \mathfrak{N}_r = \{(x, f) \in X \times B^* : \|x\|_X = \|f\| \leq r, f(\langle x, x \rangle) = \|x\|^2_2\} \]

\((r > 0)\)

and \( \mathfrak{N}_0 = \bigcup_{r > 0} \mathfrak{N}_r \), then each \( \mathfrak{N}_r (r > 0) \) and \( \mathfrak{N}_0 \) are connected subsets of \( X \times B^* \) with the norm \( \times \) weak* topology, unless \( X \) has dimension one over \( R[4] \). From now on we shall assume that \( \mathfrak{N}_0 \) has the norm \( \times \) weak* topology as a subset of \( X \times B^* \). Also we shall assume that \( X \) doesn't have dimension one over \( R \).

**Proposition 2.1.** Let \( F : \mathfrak{N}_0 \to X \) be a continuous map such that \( \|F(x, f)\|_x = \|x\|_x \) for \((x, f) \in \mathfrak{N}_0 \). Then \( z \in \Sigma^*(F) \) implies \(|z| = 1\), where \( \Sigma^*(F) \) denotes the \( B^* \)-asymptotic spectrum of \( F \in Q^*(X)[4] \).

**Proof.** Let \( z \in \Sigma^*(F) \). Then by definition of \( \Sigma^*(F) \) we can find \((x_n, f_n) \in \mathfrak{N}_0 \) such that \( \|x_n\| \geq n \) and

\[ \|z^* - F(x_n, f_n)\|_x \leq \frac{1}{n} \|x_n\|_x, \]

where \( \pi \) denotes the natural projection of \( X \times B^\tau \) onto \( X \).

Hence

\[ \|F(x_n, f_n)\|_x - \frac{1}{n} \|x_n\|_x \leq |z| \|x_n\|_x \]

\[ \leq \|F(x_n, f_n)\|_x + \frac{1}{n} \|x_n\|_x. \]

Using the assumption on \( F \),

\[ (1 - \frac{1}{n}) \|x_n\|_x \leq |z| \|x_n\|_x \leq (1 + \frac{1}{n}) \|x_n\|_x. \]

Dividing by \( \|x_n\|_x \) and letting \( n \to \infty \) completes the proof.
We note that the $B^*$-numerical range $Q^*(F)$ of $F\in W^*(X)$ is a nonempty compact connected subset of $C$, and $\Sigma^*(F)\subseteq Q^*(F)$ ($F\in Q^*(X)$) [4]. Also we recall that a Banach space $(Y,\|\cdot\|)$ is said to be uniformly convex, if whenever $x, y \in Y$, $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x+y\| \to 2$, then $\|x-y\| \to 0$.

**Proposition 2.2.** If $X$ is uniformly convex and $F\in Q^*(X)$, then $\{\lambda \in Q^*(F) : |\lambda| = |F|^*\}$ is the seminorm on $Q^*(X)$ [4].

**Proof.** Let $\lambda \in Q^*(F)$ and $|\lambda| = |F|^*$. We may assume that $\lambda \neq 0$, for otherwise $F = 0 \in Q^*(X)$, the normed space of all equivalence classes of $B^*$-quasibounded maps, i.e.,

$$Q^*(X) = Q^*(X)/N(|\cdot|^*)$$

and the result follows immediately. Since we may replace $F$ by $\lambda^{-1}F$, there is no loss of generality in assuming that $|F|^* = \lambda = 1$.

Now, there exists $(x_n, f_n) \in \Pi$ such that

$$\frac{f_n(\langle F(x_n, f_n), x_n \rangle)}{\|x_n\|_2^2 \|f_n\|} \to 1$$

as $n \to \infty$ and therefore

$$\frac{f_n(\langle (\pi + F)(x_n, f_n), x_n \rangle)}{\|x_n\|_2^2 \|f_n\|} \to 2. \quad (1)$$

Since

$$1 + \frac{\|F(x_n, f_n)\|_x}{\|x_n\|_x} = \frac{\|\pi + F\|_x}{\|x_n\|_x} \|x_n\|_x \leq \frac{\|f_n(\langle \pi + F\rangle(x_n, f_n), x_n \rangle)}{\|x_n\|_2^2 \|f_n\|}, \quad (2)$$

therefore, for every $F \in Q^*(X)$, there exists $(x_n, f_n) \in \Pi$ such that $\|x_n\|_2^2 \|f_n\| \to 2$. This implies that $\Sigma^*(F) \subseteq Q^*(F)$, and hence $Q^*(F) \subseteq \Sigma^*(F)$. Therefore, $Q^*(F)$ is a nonempty compact connected subset of $C$. 

**Remark.** The above result can be extended to the case where $X$ is a reflexive Banach space [4].
and $|F|^*$ = 1 it follows that
\[ \frac{x_n}{\|x_n\|} + \frac{F(x_n, f_n)}{\|x_n\|} \to 2. \]  
(3)

But (3) and $X$ uniformly convex imply
\[ \frac{\|\pi(F, f_n)\|}{\|x_n\|} \to 0. \]  
(4)

Hence from (4) we obtain
\[ d^*(F) = \lim \inf \frac{\|\pi(F, f_n)\|}{\|x_n\|} = 0, \]
i.e., \( 1 \in \Sigma^* F \).

On $W^*(X)$, the following seminorm is defined:\n\[ \omega^*(F) = \lim \sup \frac{|f(\langle F, f_n \rangle, x_n)|}{\|x_n\| \|f\|}. \]

\textbf{Proposition 2.3.} The multivalued function $F \in \mathcal{W}^*(X) \to \mathcal{Q}^*(F)$ is upper semicontinuous, i.e., given an neighborhood $V$ of $\mathcal{Q}^*(F)$ there exists an $\varepsilon > 0$ such that $\mathcal{Q}^*(G) \subseteq V$ for $G \in \mathcal{W}^*(X)$, $\omega^*(F - G) < \varepsilon$.

\textbf{Proof.} Suppose $\omega^*(G - F) \leq \frac{1}{n}$, $z \in \mathcal{Q}^*(G)$, $z \to z$.

We will show that $z \in \mathcal{Q}^*(F)$. It can be easily seen that this property implies the upper semicontinuity of $\mathcal{Q}^*(F)$.

By the definition of the seminorm $\omega^*(\cdot)$ we find $c > 0$ such that
\[ |f(\langle G(x, f) - F(x, f), x \rangle)| \leq \left( \frac{2}{n^2} \right) \|x\|^2 \|f\|. \]
for $(x, f) \in \mathcal{W}$, $\|x\| \geq c$. By the definition of a $\mathcal{B}^*$-numerical range we find $(x_n, f_n) \in \mathcal{W}$, $\|x_n\| \geq n + c$ such that
\[ |f_n(\langle (G - G_n)(x_n, f_n), x_n \rangle)| \leq \left( \frac{1}{n} \right) \|x_n\|^2 \|f_n\|. \]

Hence $|f_n(\langle (G - F)(x_n, f_n), x_n \rangle)| \leq |f_n(\langle (F - G_n)(x_n, f_n), x \rangle)|$.
Letting $n \to \infty$ we see that $z \in \mathcal{Q}^*(F)$.

As a consequence, the set $\{F \in \mathcal{W}^*(X) : \mathcal{Q}^*(F) \neq \phi\}$ is closed in $\mathcal{W}^*(X)$. Also the multivalued function $F \in \mathcal{Q}^*(X) \to \Sigma^*(F)$ is upper semi-continuous.

We recall that a continuous map $P : X_0 = X - \{0\} \to X$ is said to be B-numerically bounded, if the map $F : \|x\| \to X$ given by $F(x, f) = P(x)$ is B*-numerically bounded. In this case the numbers $\omega^*(F)$, $\alpha^*(F)$ and the B*-numerical range $\mathcal{Q}^*(F)$ are denoted by $\omega(P)$, $\alpha(P)$ and $\mathcal{Q}(P)$ respectively[4]. We denoted by $\mathcal{W}(X)$ the vector space of all B-numerically bounded maps on $X_0$.

Let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in $X$, and let $\phi : S \to X$ be a continuous map on $S$, i.e., a vector field on $S$. We say that $\phi$ is B-numerically bounded, if the map $\bar{\phi}(x) = \|x\| \phi(\|x\|^{-1} x)$, $x \neq 0$, is B-numerically bounded. In this case we let $\omega(\phi) = \omega(\bar{\phi})$, $\alpha(\phi) = \alpha(\bar{\phi})$ and $\mathcal{Q}(\phi) = \mathcal{Q}(\bar{\phi})$.

If we set $\Pi = \{(x, f) \in X \times \mathcal{B}^* : \|x\| = \|f\| = f(\langle x, x \rangle) = 1\}$, then $\Pi$ is a connected subset of $X \times \mathcal{B}^*$ with the norm × weak* topology[6].

**Proposition 2.4.** Let $\phi$ be a B-numerically bounded vector field on $S$. Then

(a) $\omega(\phi) = \sup_{\Pi} |g(\langle \phi(u), u \rangle)|$.

(b) $\alpha(\phi) = \inf_{\Pi} |g(\langle \phi(u), u \rangle)|$.

(c) $\mathcal{Q}(\phi) = \{g(\langle \phi(u), u \rangle) : (u, g) \in \Pi\}^*$. 
PROOF. (a) and (b) follow from
\[ f(\langle \hat{\phi}(x), x \rangle) = \frac{f(\langle \|x\|_x \phi(\|x\|_x^{-1} x), x \rangle)}{\|x\|_x^2 \|f\|} = g(\langle \hat{\phi}(u), u \rangle), \]
where \( u = \|x\|_x^{-1} x \), \( g = \|f\|^{-1} f \) and \((u, g) \in \parallel \). Now (c) becomes evident.

PROPOSITION 2.5. Let \( F \) be a continuous mapping of \( S \) into \( X \), and let \( W_{\parallel}(F) = \{f(\langle Fx, x \rangle) : (x, f) \in \parallel \} \).
Then \( W_{\parallel}(F) \) is connected.

Proof. This follows from Corollary 3.4[6].
As a consequence we see that \( \Omega(\phi) \) coincides with the closure of the \( B \)-spatial numerical range \( W_{\parallel}(\phi) \) of a continuous map \( \phi : S \rightarrow X \).

3. A nonlinear version of Lumer's formula

In [6] Yang proved the Lumer's formula
\[ \sup \text{Re } W_{\parallel}(T) = \lim_{\alpha \rightarrow 0} \frac{\|I + \alpha T\| - 1}{\alpha} \]
for any bounded linear operator \( T \) on \( X \), where \( W_{\parallel}(T) \) denotes the \( B \)-spatial numerical range of \( T \).

Our aim in this section is to prove a nonlinear version of Lumer's formula for the class of Lipschitz maps. But before we do this, we are going to state an elementary result which is a generalization of the well known properties of the logarithmic norm for bounded linear operators on a Banach space.

Lemma 3.1 [2]. Let \( Y \) be a Banach space, and let \( C(Y) \) be a vector space of continuous maps \( f : Y_{\delta} = Y - \{0\} \rightarrow Y \) such that \( I \in C(Y) \). Let \( \delta \) be a semi-norm defined on \( C(Y) \)
such that $\delta(I)=1$. If for every $f \in C(Y)$ we define

$$\delta'(f) = \lim_{\rho \to 0^+} \frac{\delta(I+\rho f)-1}{\rho} \quad (*)$$

then the limit (*) exists and satisfies the properties:
(a) $|\delta'(f)| \leq \delta(f)$.
(b) $\delta'(\mu f) = \mu \delta'(f)$, $\mu \geq 0$.
(c) $\delta'(f+g) \leq \delta'(f) + \delta'(g)$.
(d) $|\delta'(f) - \delta'(g)| \leq \delta(f-g)$.

**Lemma 3.2.** If $P \in W(X)$, then

$$\sup \Re Q(P) \leq \omega'(P). \quad (1)$$

**Proof.** From the inequality

$$\Re \frac{f(<P(x), x>)}{\|x\|_{\infty}^2 \|f\|} \leq \frac{1}{\rho} \left( \frac{|f(<x+\rho P(x), x>)|}{\|x\|_{\infty}^2 \|f\|} - 1 \right), \quad \rho > 0$$

and the obvious fact

$$\sup \Re Q(P) = \limsup_{t \to 0^-} \Re \frac{f(<P(x), x>)}{\|x\|_{\infty}^2 \|f\|},$$

we obtain

$$\sup \Re Q(P) \leq \frac{\omega(I+\rho P)-1}{\rho}, \quad \rho > 0. \quad (2)$$

Now (1) follows, if in (2) we let $\rho \to 0^+$.

On the vector space $Q(X)$ of all quasibounded maps on $X$, the following seminorm is defined:

$$|P| = \limsup_{\|x\|_{\infty} \to \infty} \frac{\|P_x\|_{[X],[3]}}{\|x\|_X}.$$

**Theorem 3.3.** If $P:X \to X$ is a Lipschitz map, i.e., there exists $k > 0$ such that
\[
\|P(x)-P(y)\| \leq k\|x-y\|, \ x, y \in X, \tag{1}
\]
then \(\sup Re \Omega(P) = \omega'(P) = \|P\|'. \tag{2}\)

Proof. Since, clearly \(\omega'(P) \leq \|P\|'\), from the previous lemma we see that it suffices to show that
\[
\|P\|' \leq \sup Re \Omega(P). \tag{3}\]
Let \(\mu = \sup Re \Omega(P)\) and \(\mu_\epsilon = \sup Re \phi_r(\|\cdot\|) \ (r > 0)\), where \(\phi_r\) is a continuous map given by
\[
\phi_r(x, f) = \frac{f(\langle P(x), x \rangle)}{\|x\|^2\|f\|}, \ (x, f) \in W_0.
\]
We have for \((x, f) \in W, (r > 0)\)
\[
\|\|I-P\|P(x)\|_x\|_{x} \geq \left| \frac{f(\langle (I-P)P(x), x \rangle)}{\|x\|^2\|f\|} \right| = 1 - \rho \left| \frac{f(\langle P(x), x \rangle)}{\|x\|^2\|f\|} \right| \geq 1 - \rho \, Re \left( \frac{f(\langle P(x), x \rangle)}{\|x\|^2\|f\|} \right) \geq 1 - \rho \sup_{r} \|P\| \geq 1 - \rho \mu,
\]
and using the fact \(\lim_{r \to 0} \mu_\epsilon = \mu\), we obtain
\[
\|\|I-P\|P(x)\|_x\|_{x} \geq 1 - \rho \mu_\epsilon > 0, \ \|x\| \geq r, \tag{4}\]
for all \(\rho > 0\) sufficiently small.

If we apply (1) we obtain
\[
\|x + \rho P(x)\| \leq \|x\| - \rho \|P(x)\| \geq \|x\| - \rho (\|P(0)\| + k\|x\|) \geq (1-k\rho)\|x\| - \rho \|P(0)\| \|x\|.
\]
Thus, if we let $0 < \rho < \frac{1}{k}$ we see from this last inequality that we can choose $\|x\|_x \geq r$ large enough so that

$$\|x + \rho P(x)\|_x \geq r.$$ 

Hence we can apply (4) with $x + \rho P(x)$ instead of $x$ and obtain

$$\|(I - \rho P)(I + \rho P)(x)\|_x \geq (1 - \rho \mu_r) \|x + \rho P(x)\|_x,$$

and

$$\|(I + \rho P)(x) - \rho P(I + \rho P)(x)\|_x \geq (1 - \rho \mu_r) \|x + \rho P(x)\|_x.$$ 

From (1) we obtain

$$\|(I + \rho P)(x) - \rho P(I + \rho P)(x)\|_x \geq \|x\|_x + \rho \|P(x)\|_x - \rho P(I + \rho P)(x)\|_x$$

$$\leq \|x\|_x + \rho \|x\|_x - (I + \rho P)(x)\|_x$$

$$= \|x\|_x + \rho^2 \|P(x)\|_x.$$ 

Thus we have

$$\|(I + \rho P)(x) - \rho P(I + \rho P)(x)\|_x \leq \|x\|_x + \rho^2 \|P(x)\|_x.$$ 

From (5) and (6) we get

$$\|x\|_x + \rho^2 \|P(x)\|_x \geq (1 - \rho \mu_r) \|x + \rho P(x)\|_x$$

and hence

$$1 + \rho^2 \frac{k \|P(x)\|_x}{\|x\|_x} \geq (1 - \rho \mu_r) \frac{\|x + \rho P(x)\|_x}{\|x\|_x}. $$

If in (7) we take the $\lim \sup$ as $r \to \infty$

we obtain $1 + \rho^2 k |P| \geq (1 - \rho \mu) |I + \rho P|,

and

$$\frac{|I + \rho P| - 1}{\rho} \leq \frac{\rho k |P| + \mu}{1 - \rho \mu}.$$ 

(8)
If in (8) we let \( p \to 0^+ \), we obtain (3), and this completes the proof.

**References**


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