ON SEMI-CLOSURE STRUCTURES AND TOPOLOGICAL MODIFYING STRUCTURES

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1. Semi-closure structures

Let $X$ be any non empty set and $\mathcal{P}(X)$ the power set of $X$. A function $u: \mathcal{P}(X) \to \mathcal{P}(X)$ is called a semi-closure structure [3] on $X$ if it satisfies the following four conditions:

1) $u(\emptyset) = \emptyset$,
2) $A \subseteq u(A)$ for each $A \in \mathcal{P}(X)$.
3) $A \cap B = u(A) \cap u(B)$ for each $A, B \in \mathcal{P}(X)$, and
4) $u(A) = u(u(A))$ for each $A \in \mathcal{P}(X)$.

A pair $(X, u)$ where $u$ is a semi-closure structure on $X$, is called a semi-closure space. These concepts are generalizations of the more familiar Kuratowski closure operator and topological spaces, respectively. For a convenience, we shall agree to use $\mathcal{U}$ as $\{A \subseteq X | u(X - A) = X - A\}$. Clearly, a semi-closure structure $u$ is satisfied i) $X, \emptyset \in \mathcal{U}$ and ii) for every $A \in \mathcal{U}$ $i \in I$, $\bigcup_{j} A \in \mathcal{U}$,

but the finite intersection of elements of $\mathcal{U}$ is not an element of $\mathcal{U}$, in general ($A$ family $\mathcal{U}$ of subsets of $X$ satisfying the above conditions i) and ii) is called a pretopology [1], a supratopology [2], or a semi-topology [6] for $X$.

The concept of semi-closure structures is motivated by the following examples:
Example 1.1. Let \((X, \mathcal{F})\) be a topological space and \(\text{-}\) and 0 denote the closure operator and the interior operator in \(X\), respectively. Then
\[
\mathcal{F}^* = \{A \subseteq X | A \subseteq A^\circ\},
\]
\[
\mathcal{F}^\circ = \{A \subseteq X | A \subseteq A^{\text{-}}\},
\]
\[
\mathcal{F}^\circ = \{A \subseteq X | A \subseteq A^{\text{-}}\}
\]
are pretopologies but not topologies for \(X[1, 5]\).

Example 1.2\([6]\). Let \(X\) and \(Y\) be any two non empty sets and \(\mathcal{F}\) a subcollection of \(\{f \mid f: X \to Y\text{ is a function}\}\). Let \(K(f, g)\) denote the coincidence set of \(f\) and \(g\), consisting of all points \(x \in X\) such that \(f(x) = g(x)\). Define \(u: \mathcal{P}(X) \to \mathcal{P}(X)\) by
\[
u(A) = \bigcap \{K(f, g) \mid K(f, g) \supseteq A, f, g \in \mathcal{F}\}.
\]
Then if \(\bigcap_{f, g \in \mathcal{F}} K(f, g) = \emptyset\), then \(u\) is a semi-closure structure on \(X\).

Moreover, if \(Y = \{0, 1\}\), then the above semi-closure structure \(u\) is a Kuratowski closure operator.

Example 1.3\([8]\). Let \(X\) be any non empty set and \(G\) and \(\mathcal{G}\) denote a transformation group of \(X\) and the equivalence relations of \(X\), respectively. Then between the complete lattice \(\mathcal{G}\) (the set of all subgroups of \(G\)) and the complete lattice \(\mathcal{G}\) there can be established a dual (inverse) Galois connection\([7]\) \(\mathcal{G} \leftrightarrow \mathcal{G}\) such that

1) \(\sigma(A) = \{a \sim b \mid f(a) = b, \text{ for some } f \in A\}\) for each subgroup \(A\) of \(G\) and

2) \(\tau(\sim) = \{f \in G \mid f(x) \sim x, \text{ for any } x \in X\}\) for each \(\sim \in \mathcal{G}\).

By the Galois connection \((\sigma \tau, \tau \sigma)\), we can prove that if
$\sigma(\phi)=\phi$ and $\tau(\phi)=\phi$, then $\sigma\tau$ and $\tau\sigma$ are semi-closure structures on $\mathcal{G}$ and $\mathcal{G}$, respectively. In [7], these structures $\sigma\tau$ and $\tau\sigma$ are called closure operators.

2. Topological modifying structures

Let $X$ be a non empty set and let $\mathcal{F}$ be a collection of semi-closure structures on $X$[3,4]. $\mathcal{F}$ is called a topological modifying structure on $X$ if for each $A,B \in \mathcal{P}(X)$ and for each $u,v \in \mathcal{F}$, there exists an element $w$ in $\mathcal{F}$ such that $u(A) \cup v(B) \supseteq w(A \cup B)$. Let $(X,u)$ be a semi-closure space. We let $\phi_u(x) = \{A \subseteq X : x \notin u(A^c)\}$. In a topological space $(X,u)$, $\phi_u(x)$ is clearly the neighborhood system at $x$ in $(X,u)$ for each $x \in X$.

Remark. (1) If a topological modifying structure $\mathcal{F}$ on $X$ has only one element $u$, then $u$ satisfies the Kuratowski closure axioms. From now on, we shall agree to use $u$ as the unique topology for $X$ determined by $u$.

(2) Any collection of semi-closure structures on $X$ is not a topological modifying structure on $X$, in general, as shown by the following example $A$.

(3) Any collection of topologies for $X$ which has at least two elements is not a topological modifying structure on $X$, in general, as shown by the following example $B$.

Example A. Let $(X,\mathcal{F})$ be a topological space and and denote the closure operator and the interior operator in $X$, respectively. Then

$$\mathcal{F}^\circ = \{A \subseteq X : A \subseteq A^0\},$$
$$\mathcal{F}^\delta = \{A \subseteq X : A \subseteq A^\circ\}$$
and
$$\mathcal{F}^\gamma = \{A \subseteq X : A \subseteq A^{-\circ}\}$$
are pretopologies \([I, 5]\), but not topologies for \(X\). If \(u\), \(v\), and \(w\) are semi-closure structures on \(X\) determined by \(\mathcal{S}^u\), \(\mathcal{S}^v\), and \(\mathcal{S}^w\), respectively, then \(\tau = \{u, v, w\}\) is not a topological modifying structure on \(X\).

**Example B.** Let \(X = \{a, b, c\}\) and \(u = \{X, \phi, \{a\}, \{a, b\}\}\) and \(v = \{X, \phi, \{b\}, \{b, c\}\}. Then u and v are topologies for \(X\) and \(u(c) \bigcup v[a] = \{a, c\} \Rightarrow u\{a, c\} = v\{a, c\} = \{a, b, c\}. Therefore \(\tau = \{u, v\}\) is not a topological modifying structure on \(X\).

**Theorem 2.1.** Let \(\tau\) be a topological modifying structure on a set \(X\). Then \(\bigcup_{x \in \tau} \phi_x(x)\) is a neighborhood system at \(x\), for each \(x \in X\). That is, \(\tau\) determines a topology \(T_\tau\) for \(X\).

**Proof.** 1) Set \(N_x = \bigcup_{x \in \tau} \phi_x(x)\) and let \(A \in N_x\). Then there is \(u \in \tau\) such that \(x \notin u(A^c)\). Since \(A^c \subset u(A^c)\), \(x \notin A^c\) and thus \(x \in A\).

2) Let \(A\) and \(B\) be two elements of \(N_x\). Then there are \(u, v \in \tau\) such that \(x \notin u(A^c)\) and \(x \notin v(B^c)\). Since \(\tau\) is a topological modifying structure on \(X\), there exists \(w \in \tau\) such that \(x \notin u(A^c) \bigcup v(B^c) \Rightarrow w(A^c \bigcup B^c) = w((A \bigcap B)^c)\). Now we have \(x \notin w((A \bigcap B)^c)\) and thus \(A \bigcap B \in N_x\).

3) Let \(A \in N_x\) and \(A \subset B \subset X\). Then there exists \(u \in \tau\) such that \(x \notin u(A^c)\). Since \(A \subset B\), \(B \subset A^c\) and \(u(B^c) \subset u(A^c)\). It follows that \(x \notin u(B^c)\) and thus \(B \in N_x\).

4) Let \(A \in N_x\). Then there exists \(u \in \tau\) such that \(x \notin u(A^c)\) and we have \(x \in X - u(A^c) \subset A\). Let \(B = X - u(A^c)\). Then we shall prove that i) \(B \in N_x\) and ii) \(A \in N_x\) for each \(y \in B\), that is, for each \(y \in B\), \(y \notin \tau(A^c)\) for some \(\tau \in \tau\).
i) Since \( u \) is a semi-closure structure on \( X \) (i.e., \( u \in \mathcal{I} \)),
\[ u(B') = u((X - u(A'))') = u(u(A')) = u(A'). \]
Since \( x \notin u(A') \),
\[ x \notin u((X - u(A'))') \] and thus \( B = X - u(A') \in N. \)

ii) Since \( B \cap u(A') = \emptyset \), \( y \notin u(A') \) and thus \( A \in N \), for each \( y \in B \).

The proof is complete.

**Theorem 2.2.** Let \( (X, \mathcal{F}) \) be a topological space and let \( \mathcal{I}' \) be a collection of semi-closure structures on \( X \) such that \( \mathcal{F} \in \mathcal{I}' \) and for each \( u \in \mathcal{I}', u \subseteq \mathcal{F} \). Then,

1. \( \mathcal{I} \) is a topological modifying structure on \( X \).
2. \( \mathcal{F} \mathcal{I} = \mathcal{F} \).

**Proof.**

1. Let \( u \) be the closure structure on \( (X, \mathcal{F}) \). For each \( v, w \in \mathcal{I} \) and for each \( A, B \in \mathcal{P}(X) \),
\[ u(A) \cup w(B) \supseteq u(A) \cup u(B) = u(A \cup B). \]
Thus \( \mathcal{I} \) is a topological modifying structure on \( X \).

2. Let \( A \) be a neighborhood of \( x \) in \( (X, \mathcal{F} \mathcal{I}) \). Then there exists an element \( v \) in \( \mathcal{I} \) such that \( x \in v (A') \). Since \( v (A) \supseteq u(B) \) for each \( A \in \mathcal{P}(X), x \notin u(A') \). Thus \( A \) is a neighborhood of \( x \) in \( (X, \mathcal{F}) \). Conversely, let \( A \) be a neighborhood of \( x \) in \( (X, \mathcal{F}) \). Then \( x \notin u(A') \) and thus \( A \in \mathcal{F} \mathcal{I}(x) \subseteq \bigcup_{\mathcal{I}} \phi_{\mathcal{I}}(x) \). Therefore \( A \) is a neighborhood of \( x \) in \( (X, \mathcal{F} \mathcal{I}) \).

The proof is complete.

**REFERENCES**


Gyeongsang National University
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