A NOTE ON WEAKLY IRRESOLUTE MAPPINGS

GYU IHN CHAE, K.K. DUBE AND O.S. PANWAR

I. Introduction.

In 1963, N. Levine introduced a class of semi-continuous mappings which properly contains the class of all continuous mappings and in [5], the notion of an irresolute mapping which is stronger than that of semi-continuity, but is independent of that of continuity was introduced. The concept of weakly irresolute mappings was introduced in [2, Definition 6]. In this note, it will be shown that the class of weakly irresolute mappings properly contains that of irresolute mapping [5], and it is independent of that of semi-continuous mappings [6], of that of almost irresolute mappings [8] and of that of set-connected mappings [3]. Further, characterizations and some basic properties of weakly irresolute mappings are investigated.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed and $f:X \to Y$ denotes a mapping from a space $X$ into a space $Y$. Let $A$ be a subset of a space $X$. By $T(X)$, $cl_X(A)$ and $int_X(A)$ ($T$, $cl(A)$ and $int(A)$ without confusions) we will denote, respectively, the topology on $X$, the closure of $A$ and the interior of $A \subseteq X$. A set $A$ is semiopen [6] in a space $X$ if there exists an $0 \in T(X)$ such that $0 \subseteq A \subseteq cl(0)$, and is semiclosed [8] iff its complement is semiopen. The intersection of all the semiclosed sets containing $A$ is called the semi-
losure [9] of A and the union of all the semiopen sets contained in \( A \subseteq X \) is called the semi-interior [9] of A. By \( SO(X) \), \( scl(A) \) and \( sint(A) \) we will denote, respectively, the family of all semiopen sets in a space \( X \), the semi-closure of A and the semi-interior of \( A \subseteq X \). It was shown in well-known papers that \( int(A) \subseteq sint(A) \subseteq A \subseteq scl(A) \subseteq cl(A) \); \( A \subseteq B \) implies \( sint(A) \subseteq sint(B) \) and \( scl(A) \subseteq scl(B) \); A is semiopen (resp. semiclosed) iff \( A = sint(A) \) (resp. \( A = scl(A) \)). A set \( N \) of a space \( X \) is called a semi-neighborhood (written semi-nbd) [1] of a point \( x \in X \) if there exists an \( U \in SO(X) \) such that \( x \in U \subseteq N \). It was shown that \( A \in SO(X) \) iff \( A \) is the semi-nbd of each of its points [1]. A point \( p \in X \) is termed a semi-limit point of \( A \) [7] iff, for each \( U \in SO(X) \) containing \( p \), \( U \cap (A \setminus \{p\}) \neq \emptyset \). The union of \( A \) and \( sd(A) \), where \( sd(A) \) denotes the set of all the semi-limit points of \( A \), called the semi-driven set of \( A \), is equal to \( scl(A) \). A is semiclosed iff \( A \) contains \( sd(A) \).

A mapping \( f: X \to Y \) is said to be irresolute [5] iff for every \( V \in SO(Y) \), \( f^{-1}(V) \in SO(X) \) iff for each \( x \in X \) and each semi-nbd \( V \) of \( f(x) \), there exists a semi-nbd \( U \) of \( x \) in \( X \) such that \( f(U) \subseteq V \). A mapping \( f: X \to Y \) is said to be semi-continuous [6] iff for every \( V \in T(Y) \), \( f^{-1}(V) \in SO(X) \). Every irresolute mapping is semi-continuous but not conversely [5]. A space \( X \) is semi-\( T_2 \) [10] iff for each pair \( x, y \in X \), \( x \neq y \), there exist disjoint \( A, B \in SO(X) \) such that \( x \in A \) and \( y \in B \) iff for each pair \( x, y \in X \), \( x \neq y \), there exists an \( U \in SO(X) \) such that \( y \in U \) and \( x \in scl(U) \). By a semi-clopen set we mean a set which is both semiopen and semiclosed. A space \( X \) is s-
WEAKLY IRRESOLUTE MAPPINGS

connected [11] iff no nonempty proper subset of \( X \) is semi-clopen; hence every indiscrete space is s-connected. A subset of a space \( X \) is s-connected iff it is s-connected as a subspace of \( X \).

II. Weakly irresolute mappings.

**Definition 1.** A mapping \( f: X \to Y \) is said to be weakly irresolute [2] if for each \( x \in X \) and each semi-nbd \( V \subseteq Y \) of \( f(x) \), there exists a semi-nbd \( U \) of \( x \) such that \( f(U) \subseteq \text{scl}(V) \).

We now give a characterization of weakly irresolute mappings.

**Theorem 1.** A mapping \( f: X \to Y \) is weakly irresolute iff for each \( 0 \in \text{SO}(Y), \subseteq \text{sint}(f^{-1}(\text{scl}(0))) \).

**Proof.** Let \( x \in f^{-1}(0) \). Then \( f(x) \in 0 \). Thus, by Definition 1, there exists a \( G \in \text{SO}(X) \) containing \( x \) such that \( f(G) \subseteq \text{scl}(0) \). This implies \( x \in G \subseteq f^{-1}(\text{scl}(0)) \), i.e., \( x \in \text{sint}(f^{-1}(\text{scl}(0))) \). Conversely, let \( x \in X \) and \( f(x) \in 0 \in \text{SO}(Y) \). Then \( x \in f^{-1}(0) \subseteq \text{sint}(f^{-1}(\text{scl}(0))) \). Let \( G = \text{sint}(f^{-1}(\text{scl}(0))) \). Then \( f(G) \subseteq \text{scl}(0) \). The proof is complete.

It is quite evident that every irresolute mapping is weakly irresolute. A weakly irresolute mapping may fail to be irresolute, as shown by the following example.

**Example 1.** Let \( X = \{a, b, c, d\} \) with \( T(X) = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\} \) and \( Y = \{p, q, r\} \) with \( T(Y) = \{\phi, Y, \{p\}\} \). Then the mapping \( f: X \to Y \), defined by \( f(a) = p, \ f(b) = f(c) = q \) and \( f(d) = r \), is, obviously, weakly irresolute but not irresolute. Note that \( f \) is also not semi-continuous.
DEFINITION 2. A space $X$ is strongly $s$-regular iff, for every point $x \in X$ and every semiclosed set $F$ of $X$ such that $x \in F$, there exist disjoint $U, V \in SO(X)$ such that $x \in U$ and $F \subseteq V$.

It can be easily shown that a space $X$ is strongly $s$-regular iff for each point $x \in X$ and each $V \in SO(X)$ containing $x$, there exists a $U \in SO(X)$ containing $x$ such that $\text{scl}(U) \subseteq V$.

THEOREM 2. Let $f: X \to Y$ be a weakly irresolute mapping. If $Y$ is strongly $s$-regular, then $f$ is irresolute and hence semi-continuous.

PROOF. Let $x \in X$ and $V \in SO(Y)$ with $f(x) \in V$. Since $Y$ is strongly $s$-regular, there exists an $M \in SO(Y)$ containing $f(x)$ such that $\text{scl}(M) \subseteq V$. Since $f$ is weakly irresolute, there exists a $U \in SO(X)$ containing $x$ such that $f(U) \subseteq \text{scl}(M) \subseteq V$. Thus $f$ is irresolute.

A semi-continuous mapping may fail to be weakly irresolute, as shown by the following example. Therefore, weakly irresolute mappings are, in general, independent of semi-continuities from Example 1.

EXAMPLE 2. Let $X = \{a, b, c\}$ with $T(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $Y = \{p, q, r\}$ with $T(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{p, q\}\}$. Then the mapping $f: X \to Y$, defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$, is semi-continuous but not weakly irresolute.

A mapping $f: X \to Y$ is said to be almost-open [13] if, for every $V \in T(Y)$, $f^{-1}(\text{cl}(V)) \subseteq \text{cl}(f^{-1}(V))$. It is known that every open mapping is almost-open and a continuous and almost-open mapping is always not open.
LEMMA 1. If $f: X \to Y$ is semi-continuous and almost-open, then $f$ is irresolute.

PROOF. It is easy to prove and is thus omitted.

From Lemma 1, we obtain that a semi-continuous mapping is weakly irresolute if it is almost-open and hence open.

THEOREM 3. If $f: X \to Y$ is weakly irresolute, then $\text{scl}(f^{-1}(V)) \subseteq f^{-1}(\text{scl}(V))$ for each $V \in SO(Y)$.

PROOF. It is sufficient to show that if $x \in \text{scl}(f^{-1}(V))$, then $x \in f^{-1}(\text{scl}(V))$. Suppose $x \in f^{-1}(\text{scl}(V))$, that is, $f(x) \in \text{scl}(V)$. Then there exists a $W \in SO(Y)$ such that $f(x) \in W$ and $W \cap Y = \emptyset$. Since $V \in SO(Y)$, we have $\text{scl}(W) \cap V = \emptyset$. Since $f$ is weakly irresolute, there exists an $U \in SO(X)$ containing $x$ such that $f(U) \subseteq \text{scl}(W)$. Accordingly, $f(U) \cap V = \emptyset$. On the other hand, if $x \in \text{scl}(f^{-1}(V))$ and $x \notin f^{-1}(V)$, then $x \in \text{scl}(f^{-1}(V))$, and so we have $U \cap f^{-1}(V) = \emptyset$ so that $f(U) \cap V \neq \emptyset$. This means a contradiction. Therefore, $x \in f^{-1}(\text{scl}(V))$. This proves the theorem.

From Theorem 3, it is obvious that if $f: X \to Y$ is weakly irresolute, then $f(\text{scl}(f^{-1}(V))) \subseteq \text{scl}(V)$ for each $V \in SO(Y)$.

DEFINITION 3. A mapping $f: X \to Y$ is termed almost irresolute [8] if for each point $x \in X$ and each semi-nbd $V \subseteq Y$ of $f(x)$, $\text{scl}(f^{-1}(V))$ is a semi-nbd of $x$.

An almost irresolute mapping need not be weakly irresolute, as show by the following example.

EXAMPLE 3. Let $X = Y = \{a, b, c, d\}$ with topologies, $T(X) = \{\emptyset, X, \{a, d\}, \{c\}, \{a, c, d\}\}$ and $T(Y) = \{\emptyset, Y, \{a\}, \{b, \}$

WEAKLY IRRESOLUTE MAPPINGS 93
94

GYU IHN CHAE, K.K. DUBE AND O.S. PANWAR

Then the mapping $f: X \to Y$, defined by $f(a) = f(b) = f(c) = a$ and $f(d) = b$, is almost irresolute but not weakly irresolute.

**Theorem 4.** If $f: X \to Y$ is almost irresolute and $\text{scl}(f^{-1}(V)) \subseteq f^{-1}(\text{scl}(V))$ for each $V \in \mathcal{SO}(Y)$, then $f$ is weakly irresolute.

**Proof.** For any point $x \in X$ and $V \in \mathcal{SO}(Y)$ containing $f(x)$, we have $\text{scl}(f^{-1}(V)) \subseteq f^{-1}(\text{scl}(V))$ by hypothesis. Since $f$ is almost irresolute, there exists a $U \in \mathcal{SO}(X)$ such that $x \in U \subseteq \text{scl}(f^{-1}(V)) \subseteq f^{-1}(\text{scl}(V))$. Thus $f(U) \subseteq \text{scl}(V)$.

The converse to Theorem 4 does not hold, in general, as shown by the following example.

**Example 4.** Let $X = Y = \{a, b, c, d\}$ with topologies, $T(X) = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $T(Y) = \{\emptyset, Y, \{a\}, \{a, c\}\}$. Then the identity mapping $i$ is weakly irresolute but not almost irresolute.

**Example 4.** An almost irresolute mapping $f: X \to Y$ is weakly irresolute iff $\text{scl}(f^{-1}(V)) \subseteq f^{-1}(\text{scl}(V))$ for each $V \in \mathcal{SO}(Y)$.

**Proof.** From Theorem 3 and 4.

**Definition 4.** Let $A$ be a subset of a space $X$. The weakly irresolute mapping from $X$ onto a subspace $A$ of $X$ is called a weakly irresolute retraction if the restriction $f|A$ is the identity mapping on $A$. We call such an $A$ a weakly irresolute retract of $X$.

**Lemma 2.** If $A$ is semiopen and $U$ is open in a space $X$, then $A \cap U$ is semiopen in $X$. (Refer to [9]).

**Theorem 5.** Let $A$ be a subset of a space $X$ and $f$: 
X→A be a weakly irresolute retraction of X onto A. If
X is T_2 then A is semiclosed in X.

Proof. Suppose A is not semiclosed. Then there exists
a semi-limit point x of A in X such that x ∈ scl(A) but
x ∉ A. Since f is weakly irresolute retraction, f(x)≠x.
Since X is T_2 there exist disjoint U, V ∈ T(X) such
that x ∈ U and f(x) ∈ V. Thus U∩cl_X(V)=∅. Also,
V∩A ∈ T(A) and hence V∩A ∈ SO(A) containing f(x).
Let W ∈ SO(X) with x ∈ W. Then U∩W ∈ SO(X) con-
tains x, by Lemma 2, and hence (U∩W)∩A ≠ ∅ because
x ∈ sd(A). Therefore, there exists a point y ∈ (U∩W)∩
A. Since y ∈ A, f(y)=y ∈ U and hence f(y) ∈ cl_X(V).
This shows that f(W)⊆cl_X(V). Now cl_A(V∩A)=cl_X(V
∩A)∩A⊆cl_X(V). Therefore, f(W)⊆cl_A(V∩A) which
implies f(W)⊆scl_A(V∩A). This contradicts the hypo-
thesis that f is weakly irresolute. Thus A is semiclosed
in X.

In Theorem 5, X is necessary Hausdorff, as shown by
the following example

Example 5. Let X={a,b,c} with an indiscrete topology
and let A={a,b} ⊂ X. Then the mapping f: X→A, de-
defined by f(a)=a, f(b)=f(c)=b, is weakly irresolute and
f|A is the identity mapping on A, that is, A is weakly
irresolute retract of X. However, A is not semiclosed
in X.

Lemma 3. A mapping f: X→Y has a semiclosed graph
G(f) [8] if for each x ∈ X, y ∈ Y such that f(x)≠y,
there exist U ∈ SO(X) and V ∈ SO(Y) containing x and
y, respectively, such that f(U)∩V=∅.
In view of the following example, a weakly irresolute mapping may fail to have a semiclosed graph.

**Example 6.** Let \( X = \{a, b, c\} \) with an indiscrete topology. Then clearly, the identity mapping \( i: X \rightarrow X \) is weakly irresolute, but \( G(i) \) is not semiclosed.

However, we have the following.

**Theorem 6.** If \( i: X \rightarrow Y \) is weakly irresolute and \( Y \) is semi-\( T_2 \) then \( G(f) \) is semiclosed in the product space \( X \times Y \).

**Proof.** Let \( x \in X \) and \( y \in Y \) such that \( y \neq f(x) \). Then there exists a \( V \in \text{SO}(Y) \) containing \( f(x) \) such that \( y \in \text{scl}(V) \), i.e., \( y \in (Y - \text{scl}(V)) \in \text{SO}(Y) \). Since \( f \) is weakly irresolute, there exists an \( U \in \text{SO}(X) \) containing \( x \) such that \( f(U) \subseteq \text{scl}(V) \). Consequently, \( f(U) \cap (Y - \text{scl}(V)) = \emptyset \) and so, \( G(f) \) is semiclosed, by Lemma 3.

The converse to Theorem 6 need not be true as shown by the following example.

**Example 7.** Let \( X = \{a, b, c\} \) be the space with \( T(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( Y = \{a, b, c\} \) be the discrete space. Then the identity mapping \( i: X \rightarrow Y \) has a semiclosed graph but not weakly irresolute.

**Lemma 4 [11].** A space \( X \) is not \( s \)-connected iff it is the union of two nonempty disjoint semiopen (respectively, semiclosed) sets.

**Theorem 7.** The \( s \)-connectedness is invariant under weakly irresolute surjections.

**Proof.** Let \( f: X \rightarrow Y \) be a weakly irresolute surjection on an \( s \)-connected space \( X \). Suppose \( Y \) is not \( s \)-connec-
Then exist nonempty disjoint $V_1, V_2 \in SO(Y)$ such that $V_1 \cup V_2 = Y$. Hence $f^{-1}(V_1) \cap f^{-1}(V_2) \neq \emptyset$ and their union is $X$. Since $f$ is surjective, $f^{-1}(V_i) \neq \emptyset$ for $i=1,2$. By Theorem 1, $f^{-1}(V_i) \subseteq \text{sint}(f^{-1}(\text{scl}(V_i)))$. Since $V_i$ is semiclosed, $f^{-1}(V_i) \subseteq \text{sint}(f^{-1}(V_i))$. Hence, $f^{-1}(V_i) \in SO(X)$ for $i=1,2$. This means that $X$ is not $s$-connected. Contradict.

Example 8. Let $X = Y = \{a, b, c\}$. Let $X$ be the indiscrete and $Y$ be the space with $T(Y) = \{\emptyset, Y, \{a, c\}, \{b\}, \{c\}\}$. Let $f: X \to Y$ be given by $f(a) = a$ and $f(b) = f(c) = b$. Then $f$ is weakly irresolute, $X$ is $s$-connected, but $f(X) = \{a, b\}$, is not $s$-connected. This example shows that the image of an $s$-connected set under a weakly irresolute mapping is not necessarily $s$-connected.

III. Weakly irresolute mappings and set-$s$-connected mappings.

Lemma 5. [3]. A mapping $f: X \to Y$ is set-$s$-connected iff for each semi-clopen subset $B$ of $f(X)$, $f^{-1}(B)$ is semi-clopen in $X$.

The following examples 9 and 10 show that the notion of weakly irresolute mappings is independent of that of set-$s$-connected mappings.

Example 9. Let $X = \{a, b, c\}$ with $T(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the mapping $f: X \to X$, defined by $f(a) = f(c) = b$ and $f(b) = c$, is weakly irresolute but not set-$s$-connected.

Example 10. Let $X = \{a, b, c, d\}$ with $T(X) = \{\emptyset, X, \{a, c\}, \{d\}, \{c\}, \{c, d\}, \{a, c, d\}\}$ and $Y = \{a, b, c\}$ with $T(Y) = \{\emptyset, Y, \{a, b, c\}\}$. Then the mapping $f: X \to Y$, defined by $f(a) = f(b) = f(c) = b$ and $f(d) = c$, is weakly irresolute but not set-$s$-connected.
\{\phi, Y, \{c\}, \{b\}, \{b, c\}\}. Then the mapping \(f:X \rightarrow Y\), defined by \(f(a) = f(d) = a\) and \(f(b) = f(c) = c\), is set-s-connected but not weakly irresolute.

**Theorem 8.** If \(f:X \rightarrow Y\) is weakly irresolute surjection, then \(f\) is set-s-connected.

**Proof.** Let \(V\) be any semi-clopen subset of \(Y\). Since \(V\) is semiclosed, \(\text{scl}(V) = V\). Thus, by Theorem 1, \(f^{-1}(V) \subseteq \text{sint}(f^{-1}(V))\). Hence \(f^{-1}(V) \in SO(X)\). Moreover, by Theorem 3, \(\text{scl}(f^{-1}(V)) \subseteq f^{-1}(V)\). Hence \(f^{-1}(V)\) is semiclosed in \(X\). Since \(f\) is surjective, Lemma 5 \(f\) is set-s-connected. It is well-known that, for every space \(X\) and each \(V \in SO(X)\), \(\text{scl}(V) \in SO(X)\) and also \(\text{cl}(V) \in SO(X)\).

**Theorem 9.** Let \(X\) and \(Y\) be spaces. If \(f:X \rightarrow Y\) is set-s-connected surjection, then \(f\) is weakly irresolute.

**Proof.** Let \(x \in X\) and \(V \in SO(Y)\) containing \(f(x)\). Then \(\text{scl}(V)\) is semi-clopen in \(Y\). Since \(f\) is set-s-connected surjection, it follows from Lemma 5 that \(f^{-1}(\text{scl}(V)) = U\) is semi-clopen in \(X\). Therefore, \(U \in SO(X)\) containing \(x\) such that \(f(U) \subseteq \text{scl}(V)\). Hence \(f\) is weakly irresolute.

**Corollary 2.** A surjection \(f:X \rightarrow Y\) is set-s-connected iff \(f\) is weakly irresolute.

**Proof.** From Theorem 8 and 9.

**Corollary 3.** If \(f:X \rightarrow Y\) is a set-s-connected surjection and \(Y\) is semi-\(T_2\) then \(G(f)\) is semiclosed in the product space \(X \times Y\).

**Proof.** From Theorem 6 and 9. In view of Example 7,
WEAKLY IRRESOLUTE MAPPINGS

the converse to Corollary 3 is not true. For, G(\xi) is semiclosed, but \xi is not set-s-connected.

Abstract

A mapping \(f: X \rightarrow Y\) is introduced to be weakly irresolute if, for each \(x \in X\) and each semi-neighborhood \(V\) of \(f(x)\), there exists a semi-neighborhood \(U\) of \(x\) in \(X\) such that \(f(U) \subseteq \text{scl}(V)\). It will be shown that a mapping \(f: X \rightarrow Y\) is weakly irresolute iff (if and only if) \(f^{-1}(V) \subseteq \text{sint}(f^{-1}(\text{scl}(V)))\) for each semiopen subset \(V\) of \(Y\). The relationship between mappings described in [3,5,6,8] and a weakly irresolute mapping will be investigated and it will be shown that every irresolute retract of a \(T_2\)-space is semiclosed.

References

[3] __________, On strongly irresolute mappings, Univ. of Ulsan Report, 16(1985), 49-
communication).


Department of Mathematics
College of Natural Science
University of Ulsan
Ulsan, KyungNam, Korea
690-00

Department of Mathematics & Stastics
University of Sagar
Sagar, M.P., India
470-003