OPTIMAL BOUNDARY CONTROL PROBLEMS
FOR SYMMETRIC HYPERBOLIC SYSTEMS
WITH CHARACTERISTIC BOUNDARIES.

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1. Introduction.

A number of authors [C.1, C.2, L.1, L.2, L.3, V.1] known to us have studied optimal boundary control problems for hyperbolic systems in several variables with noncharacteristic boundaries.

But many important problems, for instance, Maxwell equations and linearized shallow water equations, have characteristic boundary conditions.

In this paper, we study control problems for hyperbolic systems with characteristic boundaries, which is different from others.

Let \( \Omega \) be an open domain in \( \mathbb{R}^m \) with smooth boundary \( \Gamma \) for an integer \( m > 1 \). We consider a first order differential operator of the form

\[
(1.1) \quad A(x, \partial/\partial x) = \sum_{i=1}^{n} A_i(x) \frac{\partial}{\partial x_i} + C(x) \quad \text{for} \quad x \in \Omega
\]

where \( A_i(x) \) and \( C(x) \) are \((l+n) \times (l+n)\) smooth symmetric matrix valued functions on \( \bar{\Omega} \), \( l \) and \( n \) are given positive integers.

We also require that \( A_i(x) \) and \( C(x) \) are constant for sufficiently large \(|x|\) in \( \bar{\Omega} \).

We assume the uniform characteristic boundary, that is,
(1.2) the normal matrix \( N_A(x) = \sum_{i=1}^{n} A_i(x) n_i(x) \) have constant rank \( n \) for all \( x \in \Gamma \) where \( (n_i(x)) \) are the inward normals to the boundary \( \Gamma \) at \( x \) in \( \Gamma \).

Without loss of generality, we may assume that \( N_A(x) \) have \( k \) negative real eigenvalues and \( (n-k) \) positive real eigenvalues.

We now consider a mixed initial boundary value problem as

\[
\begin{align*}
\frac{\partial y}{\partial t} &= A(x, \partial/\partial x)y + h \\
\beta(x)y &= u \\
y(0) &= f
\end{align*}
\]

on \( [0, T] \times \Omega = \Omega \)

on \( [0, T] \times \Gamma = \Sigma \)

on \( \Omega \),

where \( \beta(x) \) is a boundary operator which annihilates the null space of the normal matrix \( N_A(x) \) at \( x \in \Gamma \), \( h \in L^2(\Omega) \), \( f \in L^2(\Omega) \) and \( u \in L^2(\Sigma) \).

For the simplicity, we transform our problem into one on a half-space by using local coordinate changes and a partition of unity. Thus we may assume without loss of generality that

(1.4) \( \Omega = \{ x \in \mathbb{R}^n \mid x_1 > 0 \} \) and \( \Gamma = \{ x \in \mathbb{R}^n \mid x_1 = 0 \} \).

Then the normal matrix \( N_A(x) = A_1(x) \) for \( x \) in \( \Gamma \).

By smooth change of coordinates, we may assume

(1.5) \( A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \), \( A_n = \begin{pmatrix} A_n^+ & 0 \\ 0 & A_n^- \end{pmatrix} \)

where \( A_3 \) is a negative-definite \( k \times k \) matrix and \( A_n^+ \) is positive-definite \( (n-k) \times (n-k) \) matrix.

\[*\] \( L^2(\Omega) = L^2(\Omega; \mathbb{R}^n) \), \( L^2(\Omega) = L^2(\Omega; \mathbb{R}^{l+n}) \) and \( L^2(\Sigma) = L^2(\Sigma; \mathbb{R}^l) \).
Henceforth, we assume (1.4) and (1.5) without loss of generality.
For \( x \in \mathbb{R}^n, y \in \mathbb{R}^{l+n} \), we partition \( x \) and \( y \) as \( x = (x_1, x^T) \), \( y = (y_0, y^T) \) and \( y^T = (y_1, \ldots, y_n)^T \) where \( x_1 = (x_2, \ldots, x_m)^T \), \( y_0 = (y_1, \ldots, y_l)^T \), \( y = (y_{l+1}, \ldots, y_n)^T \) and \( y^T = (y_{l+n+1}, \ldots, y_{l+n})^T \).

In order to be well-posed for the problem (1.3), it is well known that the boundary condition \( \beta(x)y = u \) can be written in the following form:

\[
y = N(x)y + u \quad \text{for} \quad x \in \mathcal{F},
\]

where \( N(x) \) is a smooth \( k \times (n-k) \) matrix valued function on \( \mathcal{F} \) which is constant for sufficiently large \( |x| \) on \( \mathcal{F} \) (see [M.1]).

We partition matrices \( A_i(x) \) as \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) where \( A_{11} \) and \( A_{22} \) are \( l \times l \) and \( n \times n \) square matrices respectively for \( j = 2, \ldots, m \), and \( A_{12} = (A_{21})^T \) is \( l \times n \) matrix.

Let us denote \( W_{11}(iw) = \sum_{j=2}^n A_{11}^{iw} \), \( W_{12}(iw) = \sum_{j=2}^n A_{12}^{iw} \), \( W_{22}(iw) = \sum_{j=2}^n A_{22}^{iw} \), for \( w = (w_2, \ldots, w_n) \in \mathbb{R}^{n-1}, w \neq 0 \).

We assume for \( W_{11}(iw) \) that

(1.7) the matrix \( W_{11}(iw) \) has distinct eigenvalues (pure imaginary) for every \( w \in \mathbb{R}^{n-1}, w \neq 0 \).

We assume, without detail, other appropriate conditions for the problem (1.3) to be well-posed in the Kreiss' sense (we refer to [M.1]).

Under appropriate assumptions [M.1], we have the following theorem.

**Theorem 1.** For any \( T > 0, f \in L^2(\Omega), h \in L^2(\Omega) \) and \( u \in L^2(\Sigma) \), the problem (1.3) has a unique strong solution \( y \)
in $L^2(Q)$ and unique strong solution $y(t)$ in $L^2(\Omega)$ at each time $t \in [0, T]$.

Moreover $y_n$ has strong boundary value in $L^2(\Sigma)$ and the following inequality holds:

$$|y(t)|_\alpha + |y|_\alpha + |y_n|_\Sigma \leq C[|f|_\alpha + |h|_\alpha + |u|_\Sigma],$$

where $C$ is a constant independent on $f$, $h$ and $u$.

**Remark.** $y_0$ in theorem 1 may not have boundary value at all in $L^2(\Sigma)$.

We are interested in boundary control problem. Hence here after it may be assumed that $f$ and $h$ are fixed, moreover let $h=0$, and $C(x)=0$.

We now are in a position to formulate optimal control problem for the system (1.3) as follows.

Suppose that $F$ and $G$ be given bounded, self-adjoint and positive-definite linear operators on $L^2(Q)$.

A quadratic functional cost $J$ is defined as

$$J(u) = |u|^2_\Sigma + (y, Fy)_\Omega + (y(T), Gy(T))_0$$

for $u \in L^2(\Sigma)$ and corresponding solution $y$ to the system (1.3).

(C.P) Our problem is to minimize the cost $J(u)$ over $u \in L^2(\Sigma)$.

Our main goal is to show that an optimal control $u^0$ exists in $L^2(\Sigma)$ and it can be synthesized as a feedback form, that is,

$$u^0(t) = BP(t)y^0(t) \quad \text{for a.e. } t \in [0, T]$$

where $B$ is an unbounded linear operator on $L^2(\Omega)$ into $L^2(\Gamma)$, $P(t)$ is a Riccati operator on $L^2(\Omega)$ and $y^0$ is the
optimal trajectory corresponding to \( u^0 \).

We state the main results in the next section and sketch briefly their proofs in section 3.

2. Main Results

We introduce an operator \( A \) on \( L^2(\Omega) \) as

\[
Ay = A(x, \partial / \partial x)y \quad \text{for} \quad y \in D(A)
\]

where the domain \( D(A) = \{ y \in L^2(\Omega) | Ay \in L^2(\Omega) \} \) and \( y_+ = Ny_+ \) on \( \Gamma \). Then it is easily seen that \( A \) is closed and densely defined on \( L^2(\Omega) \), furthermore it generates a strongly continuous semigroup \( S(t) \) on \( L^2(\Omega) \).

Remark. The adjoint \( A^* \) of \( A \) is given by

\[
(2.2) \quad A^* y = - \sum_{j=1}^{n} A^*_j(x) \frac{\partial}{\partial x_j} y - \sum_{j=1}^{n} (\frac{\partial}{\partial x_j} A_j(x)) y
\]

for \( y \in D(A^*) \) where the domain

\[
D(A^*) = \{ y \in L^2(\Omega) | A^* y \in L^2(\Omega) \}
\]

and \( y_+ = -(A^*_j)^{-1} Ny_+ \) on \( \Gamma \).

It is also well-known that \( A^* \) generates the adjoint semigroup \( S^*(t) \) of \( S(t) \) on \( L^2(\Omega) \).

In order to introduce a Dirichlet map "D" which extends boundary functions to interior functions in a certain way, we consider the following boundary value problem:

\[
(2.3) \quad \begin{cases}
A(x, \partial / \partial x)y = Ky \quad \text{on} \quad \Omega \\
y_+ = Ny_+ + u \quad \text{on} \quad \Gamma
\end{cases}
\]

for \( u \in L^2(\Gamma) \) where \( K \) is a large constant.

Then we have the following lemma.

**Lemma 2.1.**

The problem (2.3) is well-posed for a sufficiently large
number $K>0$. Moreover, the following inequality holds:

\[(2.4) \quad |y_n| + |y_s| \leq C|u| \quad \text{for} \quad u \in L^2(r)\]

where $C$ is a constant independent on $u$.

Once we have chosen $K$ so that (2.3) is well-posed, we fix $K$. For the simplicity, we may assume $K=0$ (if $K \neq 0$, we simply translate $A$ by $K$).

From lemma 2.1, we define the Dirichlet map $D$ as:

\[D_\mu = y \quad \text{if} \quad y \quad \text{is the solution to (2.3)}.\]

Then $D$ is a bounded linear operator on $L^2(r)$ into $L^2(\Omega)$.

Now have the following trace operator.

**Lemma 2.2.**

\[D^*A^*y = A^*_\mu |y|_r \quad \text{for} \quad y \in D(A^*).\]

Let us define an operator $L$ on $L^2(\mathcal{E})$ as

\[(2.5) \quad (Lu)(t) = A \int_0^t S(t-s) D u(s) ds \quad \text{for} \quad 0 \leq t \leq T\]

and $u \in D(L)$ which is a subspace of $L^2(\mathcal{E})$.

We have a semigroup representation of the solutions to the mixed problem (1.3).

**Theorem 2.3.** (Semigroup Representation).

(1) the operator $L$ is a bounded linear operator on $L^2(\mathcal{E})$ into $C([0, T]; L^2(\Omega))$.

(2) the solution $y$ to the problem (1.3) is given by

\[(2.6) \quad y(t) = S(t)f - (Lu)(t) \quad \text{for} \quad 0 \leq t \leq T.\]

From theorem 2.3, it is easily seen that our control problem $(C.P)$ has a unique optimal control $u^*$ in $L^2(\mathcal{E})$ by standard argument (see [C.1, L.1]).

Let us denote $\phi(t, s)$ the evolution operator which
describes the evolution of the optimal trajectory $y^0$ with initial time $s, 0 \leq s \leq t \leq T$ (see [C.1, L.1]). Then we have the following synthesis of optimal control $u^0$.

**Theorem 2.4.** (Feedback synthesis)

The optimal control $u^0$ can be written in the form $u^0(t) = D^*A^*P(t)y^0(t)$ a.e. $t \in [0, T]$ where $P(t)$ is a bounded self-adjoint and positive-definite operator on $L^2(\Omega)$ which satisfies the following Riccati equation:

\[
\begin{align*}
(R.E.1) & \quad \text{for } x, y \in L^2(\Omega) \text{ and } 0 \leq t \leq T, \\
(P(t)x,y)_a & = \int_t^T (\phi(t,s)x,F(t,s)y)_a \, ds \\
& + \int_t^T (D^*A^*P(s)\phi(s,t)x,D^*A^*P(s)\phi(s,t)y)_a \, ds \\
& + (\phi(T,t)x,G\phi(T,t)y)_a.
\end{align*}
\]

For a moment, we assume that

(2.7) $F$ and $G$ map $L^2(\Omega)$ into $D(A^*)$.

We denote $\mathcal{P}$ the class of one parameter families of operators $P(t)$ on $L^2(\Omega)$ which are self-adjoint, positive-definite and satisfy the following conditions;

(2.8.1) $D^*A^*P(\cdot) : L^2(\Omega) \to L^\infty([0, T]; L^2(\Omega))$ are bounded and

(2.8.2) $D^*A^*P(\cdot)S(\cdot)AD : L^2(\Gamma) \to L^2(\Sigma)$ are bounded.

**Remark.** Under assumption (2.7), it is shown that the operator $P(t)$ in theorem 2.4 is in the class $\mathcal{P}$ (see [C.1]).

Then we have the following theorem.

**Theorem 2.5.** Under assumption (2.7), the operator $P(t)$ in theorem 2.4 is the unique solution in the class $\mathcal{P}$ to the following Riccati equation
(R.E. 2) for \( x, y \in L^2(\Omega) \) and \( 0 \leq t \leq T, \)

\[
(P(t)x, y)_\alpha = \int_t^T (S(s-t)x, FS(s-t)y)_\alpha \, ds
- \int_t^T (D*A^*P(s)x, D*A^*P(s)y)_\alpha \, ds + (S(T-t)x, GS(T-t)y)_\alpha.
\]

(R.E. 3) for \( x, y \in D(A), \) a.e. \( t \) in \([0, T]\),

\[
\frac{d}{dt} (P(t)x, y)_\alpha = -(x, Fy)_\alpha - (P(t)Ax, y)_\alpha
- (P(t)x, Ay)_\alpha
+ (D*A^*P(t)x, D*A^*P(t)y),
\]

with terminal condition \( P(T) = G. \)

**Remark.** Without smoothness assumption (2.7), we are not sure whether \( D*A^*P(t)x \) are well-defined in \( L^2 \)-sense for \( x \in L^2(\Omega) \).

Suppose that the pairs \( F_n \) and \( G_n \) satisfy assumption (2.7) for all \( n=1,2,\ldots \). Let \( P_n(t) \) be the corresponding Riccati operators to the pairs \( F_n \) and \( G_n \), for \( n=1,2,3,\ldots \).

Now we do not assume any smoothness for \( F \) and \( G \). Then we have the following convergence.

**Theorem 2.6.** Suppose that \( F_n \to F \) and \( G_n \to G \) strongly on \( L^2(\Omega) \) as \( n \to \infty \). Then \( P_n(t) \to P(t) \) uniformly in \( t \in [0, T] \), strongly on \( L^2(\Omega) \) as \( n \to \infty \).

Moreover, for \( x, y \in L^2(\Omega) \), \( 0 \leq t \leq T, \)

\[
(P(t)x, y)_\alpha = \int_t^T (S(s-t)x, FS(s-t)y)_\alpha \, ds + (S(T-t)x, GS(T-t)y)_\alpha,
\]

\[
\lim_{n \to \infty} \int_t^T (D*A^*P_n(s)x, D*A^*P_n(s)y)_\alpha \, ds.
\]
3. Proofs of Results

We sketch the proofs of lemmas 2.1 and 2.2 briefly, and we omit proofs of theorems 2.3, 2.4, 2.5 and 2.6 since their proofs are similar to those in [c.1].

Suppose for a moment that the coefficients $A_1(x)$ and $N(x)$ in (2.3) are frozen at the values on a boundary point $x_0^1 \in \Gamma$.

Then we apply Fourier transform the equation (2.3) in the tangential variables $x^1$, and denote the transforms of $y$, $y_0$ and $y_n$ by $\hat{y}$, $\hat{y}_0$ and $\hat{y}_n$ respectively.

We arrive at

$$\left(3.1\right) \begin{cases} A_1 \left(0, x_0^1 \right) \frac{d \hat{y}}{dx_1} &= (K-W(0, x_0^1, iw)) \hat{y} \text{ for } x_1 > 0 \\
\hat{y} &= N(x_0^1) \hat{y}_+ - \hat{u} \text{ for } x_1 = 0 \end{cases}$$

where $W(0, x^1, iw) = \begin{bmatrix} W_{11}(0, x_0^1, iw) & W_{12}(0, x_0^1, iw) \\
W_{12}^T(0, x_0^1, iw) & W_{22}(0, x_0^1, iw) \end{bmatrix}$

and $w = (w_2, \ldots, w_m) \in \mathbb{R}^{m-1}$, $w \neq 0$.

By assumption (1.5) and (1.7), we have

$$\left(3.2\right) \begin{cases} \hat{y}_0 &= \left[ K-W_{11}(0, x_0^1, iw) \right]^{-1} W_{12}(0, x_0^1, iw) \hat{y}_n, \quad x_1 > 0 \\
A_1 \frac{d \hat{y}_n}{dx_1} &= \left[ W_{12}^T (K-W_{11})^{-1} W_{12} + (K-W_{22}) \right] \hat{y}_n, \quad x_1 > 0 \\
\hat{y}_n &= N \hat{y}_+ + \hat{u} \text{ for } x_1 = 0. \end{cases}$$

We may rewrite (3.2) in a pseudo-differential form for the variable coefficient problem as

$$\left(3.3.1\right) \hat{y}_0 = \left[ K-W_{11}(x, iw) \right]^{-1} W_{12} (x, iw) \hat{y}_n, \quad x_1 > 0$$

$$\left(3.3.2\right) \begin{cases} A_1(x) \frac{d \hat{y}_n}{dx_1} &= \left[ W_{12}^T (x, iw) (K-W_{11}(x, iw))^{-1} W_{12} \\
(x, iw) + (K-W_{22}(x, iw)) \right] \hat{y}_n, \quad x_1 > 0 \end{cases}$$
\[(3.3.3) \quad \gamma_+ = N(x^t) \gamma_+ + a, \quad x_1 = 0 \]

where \(W_{11}(x, iw), W_{12}(x, iw)\) and \(W_{22}(x, iw)\) are the pseudo-differential operators of order 1 corresponding to the differential operators \(\sum_{j=2} A_{j1}^1(x) \frac{\partial}{\partial x_j}, \quad \sum_{j=2} A_{j1}^2(x) \frac{\partial}{\partial x_j}\)

and \(\sum_{j=2} A_{j2}(x) \frac{\partial}{\partial x_j}\) respectively.

Let \(\mathcal{M}(K, x, iw) = A^{-1}_x [W_{12} (K-W_{11})^{-1} W_{12} + K-W_{22}]\).

Then we arrive at
\[(3.4.1) \quad \gamma_0 = (K-W_{11})^{-1} W_{12} \gamma_+ \quad x_1 > 0\]
\[(3.4.2) \quad \frac{d \gamma_+}{dx_1} = \mathcal{M} \gamma_+ \quad x_1 > 0\]
\[(3.4.3) \quad \gamma_+ = N \gamma_+ + a \quad x_1 = 0.\]

The problems
\[(3.4.2)\] and \[(3.4.3)\] are the same kind Majda and Osher studied in \([M.1]\).

That is, they showed that there exists a symmetrizer of \(\mathcal{M}\) whose symbol \(\bar{R}(K, x, iw)\) is of order zero and satisfies the following properties (see \([M.1]\));

\[(3.5.1) \quad \bar{R} \text{ is Hermitian,}\]
\[(3.5.2) \quad v^T \bar{R} v \geq \delta |v|^2 - \epsilon |g|^2 \quad \text{for all vectors satisfying the boundary condition } v_+ = N v_+ + g,\]
\[(3.5.3) \quad \text{Re} (\bar{R} \mathcal{M}) \geq \delta \text{ where } \delta > 0 \text{ and } \epsilon > 0 \text{ are constants independent on } x \in \Omega, \quad w \in \mathbb{R}^{n-1} \text{ and } K > 0 \text{ large enough.}\]

Thus combining \((3.4.2)\) and \((3.4.3)\) with \((3.5.1)\), \((3.5.2)\) and \((3.5.3)\), we have, for sufficiently large \(K > 0\),

\[(3.6) \quad |y_+|_\alpha + |y_+|_r \leq C|u|_r \quad \text{where } C \text{ is a constant independent on } u.\]

On the other hand, we suppose that \(y\) is a solution to \((2.3)\).
We take the inner product (2.3) with $y$ on $\Omega$. Then we have

$$\tag{3.7} (A(x, \partial/\partial x)y, y)_n = K|y|^2_n.$$ 

By Green's formula, the left hand side of (3.7) becomes

$$-\frac{1}{2} (A_n y_n, y_n)_r - \frac{1}{2} \langle y, \sum_{i=1}^m \left( \frac{\partial}{\partial x_i} A_i \right)y \rangle_n.$$ 

That is,

$$K|y|^2_n + \frac{1}{2} \langle y, \sum_{i=1}^m \left( \frac{\partial}{\partial x_i} A_i \right)y \rangle_n = -\frac{1}{2} (A_n y_n, y_n)_r.$$ 

Thus, for sufficiently large $K > 0$, we arrive at

$$\tag{3.8} |y|_n \leq C |y|_r$$ 

for a constant $C$.

From (3.6) and (3.8), we derive the inequality $|y|_n + |y|_r \leq C|u|_r$ for sufficiently large $K > 0$.

Once we have the energy inequality, we can deduce the uniqueness and existence easily as in [C.1].

This completes the proof of lemma 2.1.

We assume, without loss of generality, $K = 0$.

From Green's formula, we have for $y \in D(A^*)$ and $g \in L^2(\Gamma)$,

$$\tag{3.9} (A^*y, Dg)_a = \langle y, A(x, \partial/\partial x) Dg \rangle_a + \langle A_n y_n, (Dg)_n \rangle_r.$$ 

By the definition of the operator $D$, (3.9) becomes

$$(A^*y, Dg) = (A_n y_n, (Dg)_n)_r.$$ 

From the fact that $y \in D(A^*)$ and $(Dg)_r = N(Dg)_r + g$ on $\Gamma$, we arrive at

$$(A^*y, Dg)_a = (A_n y_n, g)_r$$ 

which implies lemma 2.2.
References


