EXTREMELY MEASURABLE SUBALGEBRAS

S.K. AYYASWAMY

1. Notations and preliminaries:

For each \( a \in S \) and \( f \in m(S) \), denote by \( l_\alpha f(s) = f(as) \) for all \( s \in S \). If \( A \) is a norm closed left translation invariant subalgebra of \( m(S) \) (i.e. \( l_\alpha f \in A \) whenever \( f \in A \) and \( a \in S \)) containing \( 1 \), the constant one function on \( S \) and \( \varphi \in A^* \), the dual of \( A \), then \( \varphi \) is a mean on \( A \) if \( \varphi(f) \geq 0 \) for \( f \geq 0 \) and \( \varphi(1) = 1 \), \( \varphi \) is multiplicative if \( \varphi(fg) = \varphi(f) \varphi(g) \) for all \( f, g \in A \); \( \varphi \) is left invariant if \( \varphi(1sf) = \varphi(f) \) for all \( s \in S \) and \( f \in A \). It is well known that the set of multiplicative means on \( m(S) \) is precisely \( \beta S \), the Stone-Cech compactification of \( S \).[7]

A subalgebra of \( m(S) \) is (extremely) left amenable, denoted by \((ELA) LA\) if it is norm closed, left translation invariant containing constants and has a multiplicative left invariant mean (LIM). A semigroup \( S \) is \((ELA) LA\), if \( m(S) \) is \((ELA) LA\).

A subset \( E \subseteq S \) is left thick (T. Mitchell, [4]) if for any finite subset \( F \subseteq S \), there exists \( s \in S \) such that \( Fs \subseteq E \) or equivalently, the family \( \{s^{-1}E : s \in S \} \) has finite intersection property.

2. Characterisation of extremely measurable subalgebras:

**Definition 2.1.** Let \( S \) be a semigroup and \( \mathcal{I} \) a subalgebra of \( P(S) \), the power set of \( S \), such that \( aE \in \mathcal{I} \) whenever \( E \in \mathcal{I} \) and \( a \in S \). \( \mathcal{I} \) is left measurable if there exists a non-negative finitely additive measure \( \mu \) on \( \mathcal{I} \) such that \( \mu(aE) = \mu(E) \) for all \( E \in \mathcal{I} \) and \( a \in S \).

Let \( \mathcal{I} \) be a subalgebras of \( P(S) \) such that \( a^{-1}E \) and \( aE \in \mathcal{I} \) whenever \( E \in \mathcal{I} \) and \( a \in S \). Let \( m(\mathcal{I}, S) \) denote the norm closure of the linear span of the set \( \{1_E \cap E \in \mathcal{I} \} \). Then \( m(\mathcal{I}, S) \) is a norm closed subalgebra of \( m(S) \) containing constants. Let \( \Delta(\mathcal{I}, S) \) denote the set of all multiplicative means on \( m(\mathcal{I}, S) \).

**Remark 2.2.** There are proper subalgebras of \( P(S) \) satisfying the condition that both \( a^{-1}E \) and \( aE \) are elements of the subalgebra whenever \( E \) is its element. Examples:

1. Consider the right zero semigroup \( N \) of natural numbers (i.e.) \( a \cdot b = b \) for all \( a, b \in N \). Let \( \mathcal{I} \) be the algebra of those subsets of \( N \) which are either finite
or their complements are finite. It is clear that $\mathcal{I}$ is a proper subalgebra of $m(S)$ and that $sE=E=s^{-1}E$ for any $s \in N$ and $E \subset N$.

(2) If $S$ is a group, consider the collection $\mathcal{I}=\{\phi, S\}$. $\mathcal{I}$ is trivially a proper subalgebra of $m(S)$ satisfying the condition that $s^{-1}E$ and $sE \in \mathcal{I}$ whenever $E \in \mathcal{I}$ and $s \in S$.

**Definition 2.3.** Let $\mathcal{I}$ be a subalgebra of $P(S)$ such that $a^{-1}E$ and $aE \in \mathcal{I}$ whenever $E \in \mathcal{I}$ and $a \in S$. A mean $\mu$ on $m(\mathcal{I}, S)$ is called an extremely left invariant measure if $\mu(1_E)=\mu(1_{aE})$ for every $a \in S$ and $E \in \mathcal{I}$. $m(\mathcal{I}, S)$ is called extremely left measurable if there exists an extremely left invariant measure on $m(\mathcal{I}, S)$.

**Remark 2.4.** (1) when $\mathcal{I}=P(S)$, Definition 2.3 is consistent with P.V. Ramakrishnan's concept of extremely left invariant measures on $S[5]$.

(2) J. Sorenson has proved in [6] that every left invariant measure on a Semigroup $S$, if exists, gives rise to a left invariant mean on $m(S)$. The corresponding result for $m(\mathcal{I}, S)$ also holds. That is, every extremely left invariant measure on $m(\mathcal{I}, S)$ gives rise to a left invariant mean on $m(\mathcal{I}, S)$. (The proof adapts verbatim to that of J. Sorenson's result and so we omit). Converse is not true even for $\mathcal{I}=P(S)$. For example, consider the semigroup, $S=G \cup \{0\}$ where $G$ is a finite group and $0$ is a new element added to $G$ in such a way that the new multiplication of elements is as follows: $g \cdot 0=0 \cdot g=0$ for all $g \in G$ and $g_1 \cdot g_2$ is the original product defined in $G$. Then the function $\mu$ on $m(S)$ defined by $\mu(f)=f(0)$ is a multiplicative left invariant mean but it is not a left invariant measure. However, for cancellative semigroups, the converse is true.

**Proposition 2.5.** Let $S$ be a cancellative semigroup such that $m(\mathcal{I}, S)$ admits a LIM. Then every LIM on $m(\mathcal{I}, S)$ is a left invariant measure on $m(\mathcal{I}, S)$.

**Proof.** $S$ is cancellative implies that $s^{-1}(sA)=A$ for all $s \in S$ and $A \subset S$. Therefore, if $\mu$ is a LIM on $m(\mathcal{I}, S)$ and $1_A \in m(\mathcal{I}, S)$, then $\mu(1_A)=\mu(1_{s^{-1}(sA)})=\mu(1_{sA})=\mu(1_A)$.

In Theorem 2.6, we characterise all those subalgebras of the form $m(\mathcal{I}, S)$ which are extremely left measurable. The proof follows the pattern of A.T. Lau ([3], Theorem 2)

**Theorem 2.6.** Let $S$ be a semigroup and $\mathcal{I}$ an algebra of subsets of $S$ such that $s^{-1}E$ and $sE \in \mathcal{I}$ whenever $E \in \mathcal{I}$ and $s \in S$. Consider the following statements:

(A) $m(\mathcal{I}, S)$ is extremely left measurable.

(B) For each finite collection $\{E_1, E_2, \cdots, E_n\}$ of disjoint sets from $\mathcal{I}$ with union $S$, there exists at least one $E_i$ such that $\bigcap_{i=1}^{n} sE_i \neq \phi$ where closure is taken in $\Delta(\mathcal{I}, S)$

In general (A) implies (B). If $S$ is a left cancellative semigroup then (B)
implies (A).

Proof. (A) implies (B)

Let \( \varphi \) be a left invariant measure on \( m(\mathcal{F}, S) \). Then we have \( 1 = \varphi(S) = \sum_{i=1}^{\infty} \varphi(E_i) \). Hence \( \varphi(E_i) > 0 \) for some \( i \) which implies that \( \varphi(sE_i) = \varphi(E_i) = 1 \) since \( \varphi \) is multiplicative. Consequently \( \varphi \in \mathcal{E} \cap \{ \cap_{i \in S} E_i \} \) (closure in \( \mathcal{A}(\mathcal{F}, S) \)). That is, \( \mathcal{E} \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \).

(B) implies (A)

Let \( \mathcal{D} \) be the set each of whose elements is a finite collection \( \{E_1, \ldots, E_n\} \) of disjoint sets in \( \mathcal{F} \) with union \( S \). \( \mathcal{D} \) can be made as a directed set by defining the order \( P_1 \geq P_2 \) to mean that each set in \( P_2 \) is the union of sets in \( P_1 \) where \( P_1 \) and \( P_2 \in \mathcal{D} \). For each \( E \in \mathcal{F} \) define \( \mathcal{K}_E = \{ \varphi \in \mathcal{A}(\mathcal{F}, S) : \varphi(sE) = \varphi(E) \) for all \( s \in S \}. \)

If we can show that the collection \( \{ \mathcal{K}_E : E \in \mathcal{F} \} \) is a collection of nonempty closed subsets of \( \mathcal{A}(\mathcal{F}, S) \) with finite intersection property, then by the compactness of \( \mathcal{A}(\mathcal{F}, S) \) it will follow that \( \cap_{E \in \mathcal{F}} \mathcal{K}_E \neq \varnothing \). Then any \( \varphi \in \cap_{E \in \mathcal{F}} \mathcal{K}_E \) will be a multiplicative left invariant measure on \( m(\mathcal{F}, S) \).

Clearly each \( \mathcal{K}_E \) is closed in \( \mathcal{A}(\mathcal{F}, S) \). We shall prove that for each \( E \in \mathcal{F} \), \( \mathcal{K}_E \) is nonempty: If \( E \in \mathcal{F} \), we have \( \{E, E'\} \in \mathcal{D} \) where \( E' \) denotes the complement of \( E \) in \( S \). Therefore, by the condition (B), either \( E \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \) or \( E' \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \). In case, \( E \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \) then any \( \varphi \in \mathcal{E} \cap \{ \cap_{i \in S} E_i \} \) is an element of \( \mathcal{K}_E \) since \( \varphi(sE) = \varphi(E) = 1 \). On the other hand, if \( E' \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \) then for any \( \varphi \in \mathcal{E} \cap \{ \cap_{i \in S} E_i \} \) we have \( \varphi(sE') = \varphi(E') = 1 \). As \( S \) is left cancellative, \( sE' \cap sE = \varnothing \) for all \( s \in S \) and therefore \( \varphi(sE') = \varphi(E') = 0 \). That is \( \varnothing \in \mathcal{K}_E \).

To show that the family \( \{ \mathcal{K}_E : E \in \mathcal{F} \} \) has finite intersection property, let \( E_1, \ldots, E_n \) be a finite number of arbitrary elements in \( \mathcal{F} \). For each \( i = 1, 2, \ldots, n \), let \( P_i = \{E\in E_i'\} \) and choose \( \mathcal{P}_o \in \mathcal{D} \) such that \( \mathcal{P}_o \geq P_i \) for each \( i, 1 \leq i \leq n \). By assumption, there exists \( \mathcal{F} \) in \( \mathcal{P}_o \) such that \( \mathcal{F} \cap \{ \cap_{i \in S} E_i \} \neq \varnothing \). Let \( \varphi_o \in \mathcal{A}(\mathcal{F}, S) \) such that \( \varphi_o(sF) = \varphi_o(F) = 1 \) for all \( s \in S \). If \( \mathcal{F} \subset E_i \) then \( sF \subset sE_i \) for all \( s \in S \) so that \( \varphi_o(sE_i) = \varphi_o(E_i) = 1 \) for all \( s \in S \). If \( \mathcal{F} \subset E_i' \), then \( \varphi_o(sE_i') = \varphi_o(E_i') = 1 \). Therefore, by the left cancellation of \( S \), it follows, as in the previous paragraph, that \( \varphi_o(sE_i) = \varphi_o(E_i) = 0 \). Hence \( \varnothing \subset \sum_{i=1}^{n} \mathcal{K}_E_i \). This completes the proof.

Remark 2.7. If the condition 'left cancellation' of \( S \) is removed, then (B) implies (A) need not be true. The example \( S = G \cup \{0\} \) in Remark 2.5 is a nonleft cancellative semigroup which is not left measurable. However, since \( S \) is \( ELA \) for each finite collection \( \{E_i, \ldots, E_n\} \) of disjoint sets in \( S \) with union \( S \), there exists at least one \( E_i \) which is left thick ([3], Theorem 2): This set \( E_i \) is precisely that set which contains the element 0. Therefore, \( E_i \cap \{ \cap_{i \in S} E_i \} = \{ 0 \} \neq \varnothing \).
As $m(\mathcal{F}, S)$ is a Banach algebra containing identity, by a standard argument in harmonic analysis, the extreme left measurability of $m(\mathcal{F}, S)$ can be characterized in terms of the ideal $A$ spanned by $\{1_{aE} - 1_E : E \in \mathcal{F} \text{ and } a \in S\}$. Theorem 2.8 gives three equivalent conditions.

**Theorem 2.8.** Let $S$ be a semigroup and $\mathcal{F}$ a subalgebra of subsets of $S$ such that $a^{-1}E$ and $aE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $a \in S$. Then the following are equivalent:

(i) $m(\mathcal{F}, S)$ is extremely left measurable.

(ii) $A$ is not uniformly dense in $m(\mathcal{F}, S)$.

(iii) $\inf \{\|1 - h\| : h \in A\} > 0$

**Proof.** (i) implies (ii). Let $m(\mathcal{F}, S)$ be extremely left measurable with a left invariant measure. Then $\mu(f) = 0$ for all $f \in A$. Therefore, since $m(\mathcal{F}, S)$ is a Banach algebra containing identity, there exists a proper maximal ideal containing $A$. Hence $A$ is not uniformly dense in $m(\mathcal{F}, S)$.

(ii) implies (iii). Since $A$ is not uniformly dense in $m(\mathcal{F}, S)$, $1 \in A$. $1 \in \bar{A}$ also where $\bar{A}$ is the closure of $A$ in $\mathcal{A}(\mathcal{F}, S)$. Hence $\inf \{\|1 - h\| : h \in A\} > 0$.

(iii) implies (i). If $\inf \{\|1 - h\| : h \in A\} > 0$, then $A$ is not dense in $m(\mathcal{F}, S)$. Therefore there exists a multiplicative linear functional $\mu$ on $m(\mathcal{F}, S)$ which vanishes on $A$. This $\mu$ is clearly a multiplicative left invariant mean such that $\mu(1_{aE}) = \mu(1_E)$ for all $E \in \mathcal{F}$ and $a \in S$.

**References**


The School of Mathematics, Madurai Kamaraj University, Madurai–625 021, INDIA