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EXTREMELY MEASURABLE SUBALGEBRAS

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1. Notations and preliminaries:

For each $a \in S$ and $f \in m(S)$, denote by $l_a f(s) = f(as)$ for all $s \in S$. If A is a norm closed left translation invariant subalgebra of $m(S)$ (i.e. $l_a f \in A$ whenever $f \in A$ and $a \in S$) containing 1, the constant one function on S and $\varphi \in A^*$, the dual of A, then φ is a mean on A if $\varphi(f) \geq 0$ for $f \geq 0$ and $\varphi(1) = 1$, φ is multiplicative if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in A$; φ is left invariant if $\varphi(1_S f) = \varphi(f)$ for all $s \in S$ and $f \in A$. It is well known that the set of multiplicative means on $m(S)$ is precisely βS, the Stone-Cěch compactification of S[7].

A subalgebra of $m(S)$ is (extremely) left amenable, denoted by $(ELA)LA$ if it is nom closed, left translation invariant containing contants and has a multiplicative left invariant mean (LIM). A semigroup S is (ELA) LA, if $m(S)$ is $(ELA)LA$.

A subset $E \subset S$ is left thick (T. Mitchell, [4]) if for any finite subset $F \subset S$, there exists $s \in S$ such that $Fs \subset E$ or equivalently, the family $\{s^{-1}E : s \in S\}$ has finite intersection property.

2. Characterisation of extremely measurable subalgebras:

DEFINITION 2.1. Let S be a semigroup and $\mathcal{F}$ a subalgebra of $P(S)$, the power set of S, such that $aE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $a \in S$. $\mathcal{F}$ is left measurable if there exists a non-negative finitely additive measure μ on $\mathcal{F}$ such that $\mu(aE) = \mu(E)$ for all $E \in \mathcal{F}$ and $a \in S$.

Let $\mathcal{F}$ be a subalgebras of $P(S)$ such that $a^{-1}E$ and $aE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $a \in S$. Let $m(\mathcal{F}, S)$ denote the norm closure of the linear span of the set $\{1_E : E \in \mathcal{F}\}$. Then $m(\mathcal{F}, S)$ is a norm closed subalgebra of $m(S)$ containing constants. Let $\Delta(\mathcal{F}, S)$ denote the set of all multiplicative means on $m(\mathcal{F}, S)$.

REMARK 2.2. There are proper subalgebras of $P(S)$ satisfying the condition that both $a^{-1}E$ and aE are elements of the subalgerba whenver E is its element. Examples:-

(1) Consider the right zero semigroup N of natural numbers (i.e.) $a.b. = b$ for all $a, b \in N$. Let $\mathcal{F}$ be the algebra of those subsets of N which are either finite

or their complements are finite. It is clear that $\mathcal{F}$ is a proper subalgebra of $m(S)$ and that $sE=E=s^{-1}E$ for any $s \in N$ and $E \subset N$.

(2) If S is a group, consider the collection $\mathcal{F}=\{\phi, S\}$. $\mathcal{F}$ is trivially a proper subalgebra of $m(S)$ satisfying the condition that $s^{-1}E$ and $sE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $s \in S$.

DEFINITION 2.3. Let $\mathcal{F}$ be a subalgebra of $P(S)$ such that $a^{-1}E$ and $aE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $a \in S$. A mean μ on $m(\mathcal{F}, S)$ is called an extremely left invariant measure if $\mu(1_E)=\mu(1_{aE})$ for every $a \in S$ and $E \in \mathcal{F}$. $m(\mathcal{F}, S)$ is called extremely left measurable if there exists an extremely left invariant measure on $m(\mathcal{F}, S)$.

REMARK 2.4. (1) when $\mathcal{F}=P(S)$, Definition 2.3 is consistent with P. V. Ramakrishnan's concept of extremely left invariant measures on S[5].

(2) J. Sorenson has proved in [6] that every left invariant measure on a Semigroup S, if exists, gives rise to a left invariant mean on $m(S)$. The corresponding result for $m(\mathcal{F}, S)$ also holds. That is, every extremely left invariant measure on $m(\mathcal{F}, S)$ gives rise to a left invariant mean on $m(\mathcal{F}, S)$. (The proof adapts verbatim to that of J. Sorenson's result and so we omit). Converse is not true even for $\mathcal{F}=P(S)$. For example, consider the semigroup, $S=G \cup \{0\}$ where G is a finite group and 0 is a new element added to G in such a way that the new multiplication of elements is as follows: $g \cdot 0=0 \cdot g=0$ for all $g \in G$ and $g_1 \cdot g_2$ is the original product defined in G. Then the function μ on $m(S)$ defined by $\mu(f)=f(0)$ is a multiplicative left invariant mean but it is not a left invariant measure. However, for cancellative semigroups, the converse is true.

PROPOSITION 2.5. *Let S be a cancellative semigroup such that $m(\mathcal{F}, S)$ admits a LIM. Then every LIM on $m(\mathcal{F}, S)$ is a left invariant measure on $m(\mathcal{F}, S)$.*

Proof. S is cancellative implies that $s^{-1}(sA)=A$ for all $s \in S$ and $A \subset S$. Therefore, if μ is a LIM on $m(\mathcal{F}, S)$ and $1_A \in m(\mathcal{F}, S)$, then $\mu(1_A)=\mu(1_s-1_{(sA)})=\mu(1_s 1_{sA})=\mu(1_{sA})$.

In Theorem 2.6, we characterise all those subalgebras of the form $m(\mathcal{F}, S)$ which are extremely left measurable. The proof follows the pattern of A. T. Lau ([3], Theorem 2)

THEOREM 2.6. *Let S be a semigroup and $\mathcal{F}$ an algebra of subsets of S such that $s^{-1}E$ and $sE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $s \in S$. Consider the following statements:*

(A) *$m(\mathcal{F}, S)$ is extremely left measurable.*

(B) *For each finite collection $\{E_1, E_2, \cdots, E_n\}$ of disjoint sets from $\mathcal{F}$ with union S, there exists at least one E_i such that $\bar{E}_i \cap \{\bigcap_{s \in S} \overline{sE_i}\} \neq \phi$ where closure is taken in $\Delta(\mathcal{F}, S)$*

In general (A) *implies* (B). *If S is a left cancellative semigroup then* (B) *implies* (A).

Proof. (A) implies (B)

Let φ be a left invariant measure on $m(\mathcal{F}, S)$. Then we have $1=\varphi(S)=\sum_{i=1}^{n}\varphi(E_i)$. Hence $\varphi(E_i)>0$ for some i which implies that $\varphi(sE_i)=\varphi(E_i)=1$ since φ is multiplicative. Consequently $\varphi\in\bar{E}_i\cap\{\bigcap_{s\in S}\overline{sE_i}\}$ (closure in $\Delta(\mathcal{F}, S)$. That is, $\bar{E}_i\cap\{\bigcap_{s\in S}\overline{sE_i}\}\neq\phi$.

(B) implies (A)

Let $\mathcal{D}$ be the set each of whose elements is a finite collection $\{E_1,\cdots,E_n\}$ of disjoint sets in $\mathcal{F}$ with union S. $\mathcal{D}$ can be made as a directed set by defining the order $P_1\geq P_2$ to mean that each set in P_2 is the union of sets in P_1 where P_1 and $P_2\in\mathcal{D}$. For each $E\in\mathcal{F}$ define $K_E=\{\varphi\in\Delta(\mathcal{F}, S):\varphi(sE)=\varphi(E)$ for all $s\in S\}$. If we can show that the collection $\{K_E : E\in\mathcal{F}\}$ is a collection of nonempty closed subsets of $\Delta(\mathcal{F}, S)$ with finite intersection property, then by the compactness of $\Delta(\mathcal{F}, S)$ it will follow that $\bigcap_{E\in\mathcal{F}}K_E\neq\phi$. Then any $\varphi\in\bigcap_{E\in\mathcal{F}}K_E$ will be a multiplicative left invariant measure on $m(\mathcal{F}, S)$.

Clearly each K_E is closed in $\Delta(\mathcal{F}, S)$. We shall prove that for each $E\in\mathcal{F}$, K_E is nonempty: If $E\in\mathcal{F}$, we have $\{E, E'\}\in\mathcal{D}$ where E' denotes the complement of E in S. Therefore, by the condition (B), either $\bar{E}\cap\{\bigcap_{s\in S}\overline{sE}\}\neq\phi$ or $\bar{E}'\cap\{\bigcap_{s\in S}\overline{sE'}\}\neq\phi$. In case, $\bar{E}\cap\{\bigcap_{s\in S}\overline{sE}\}\neq\phi$ then any $\varphi\in\bar{E}\cap\{\bigcap_{s\in S}\overline{sE}\}$ is an element of K_E since $\varphi(sE)=\varphi(E)=1$. On the other hand, if $\bar{E}'\cap\{\bigcap_{s\in S}\overline{sE'}\}\neq\phi$ then for any $\psi\in\bar{E}'\cap\{\bigcap_{s\in S}\overline{sE'}\}$ we have $\psi(sE')=\psi(E')=1$. As S is left cancellative, $sE'\cap sE=\phi$ for all $s\in S$ and therefore $\psi(sE)=\psi(E)=0$. That is $\psi\in K_E$.

To show that the family $\{K_E ; E\in\mathcal{F}\}$ has finite intersection property, let $E_1,\cdots, E_n$ be a finite number of arbitrary elements in $\mathcal{F}$. For each $i=1,2,\cdots,n$, let $P_i=\{E_i, E_i'\}$ and choose $P_o\in\mathcal{D}$ such that $P_o\geq P_i$ for each i, $1\leq i\leq n$. By assumption, there exists F in P_o such that $\bar{F}\cap\{\bigcap_{s\in S}\overline{sF}\}\neq\phi$. Let $\varphi_0\in\Delta(\mathcal{F}, S)$ such that $\varphi_0(sF)=\varphi_0(F)=1$ for all $s\in S$. If $F\subset E_i$ then $sF\subset sE_i$ for all $s\in S$ so that $\varphi_o(sE_i)=\varphi_o(E_i)=1$ for all $s\in S$. If $F\subset E_i'$, then $\varphi_o(sE_i')=\varphi_o(E_i')=1$. Therefore, by the left cancellation of S, it follows, as in the previous paragraph, that $\varphi_o(sE_i)=\varphi_o(E_i)=0$. Hence $\varphi_o\in\sum_{i=1}^{n}K_{E_i}$. This completes the proof.

REMARK 2.7. If the condition 'left cancellation' of S is removed, then (B) implies (A) need not be true. The example $S=G\cup\{0\}$ in Remark 2.5 is a nonleft cancellative semigroup which is not left measurable. However, since S is *ELA* for each finite collection $\{E_i,\cdots,E_n\}$ of disjoint sets in S with union S, there exists at least one E_i which is left thick ([3], Theorem 2) : This set E_i is precisely that set which contains the element 0. Therefore, $E_i\cap\{\bigcap_{s\in S}sE_i\}=\{0\}\neq\phi$.

As $m(\mathcal{F}, S)$ is a Banach algebra containing identity, by a standard argument in harmonic analysis, the extreme left measurability of $m(\mathcal{F}, S)$ can be characterized in terms of the ideal A spanned by $\{1_{aE}-1_E : E \in \mathcal{F}$ and $a \in S\}$. Theorem 2.8 gives three equivalent conditions.

THEOREM 2.8. *Let S be a semigroup and $\mathcal{F}$ a subalgebra of subsets of S such that $a^{-1}E$ and $aE \in \mathcal{F}$ whenever $E \in \mathcal{F}$ and $a \in S$. Then the following are equivalent:*

(i) *$m(\mathcal{F}, S)$ is extremely left measurable.*
(ii) *A is not uniformly dense in $m(\mathcal{F}, S)$.*
(iii) *Inf* $\{\|1-h\| : h \in A\} > 0$

Proof. (i) implies (ii). Let $m(\mathcal{F}, S)$ be extremely left measurable with a left invariant measure. Then $\mu(f)=0$ for all $f \in A$. Therefore, since $m(\mathcal{F}, S)$ is a Banach algebra containing identity, there exists a proper maximal ideal containing A. Hence A is not uniformly dense in $m(\mathcal{F}, S)$.

(ii) implies (iii). Since A is not uniformly dense in $m(\mathcal{F}, S)$, $1 \in A$. $1 \in \bar{A}$ also where $\bar{A}$ is the closure of A in $\Delta(\mathcal{F}, S)$. Hence inf $\{\|1-h\| : h \in A\} > 0$.

(iii) implies (i). If Inf $\{\|1-h\| : h \in A\} > 0$, then A is not dense in $m(\mathcal{F}, S)$. Therefore there exists a multilicative linear functional μ on $m(\mathcal{F}, S)$ which vanishes on A. This μ is clearly a multiplicative left invariant mean such that $\mu(1_{aE})=\mu(1_E)$ for all $E \in \mathcal{F}$ and $a \in S$.

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