Bull. Korean Math. Soc. **22**(1985), No. 1, pp. 11-15

CLASSIFICATION OF CLIFFORD ALGEBRAS OF FREE QUADRATIC SPACES OVER FULL RINGS

JAE-GYEOM KIM

1. Introduction and preliminaries

Mandelberg [9] has shown that a Clifford algebra of a free quadratic space over an arbitrary semi-local ring R in Brawer-Wall group $BW(R)$ is determined by its rank, determinant, and Hasse invariant. In this paper, we prove a corresponding result when R is a full ring.

Throughout this paper, unless otherwise specified, we assume that R is a commutative ring having 2 a unit. A quadratic space (V, B, ϕ) over R is a finitely generated projective R-module V with a symmetric bilinear mapping $B: V\times V\to R$ which is non-degenerate (i.e., the natural mapping $V\to\mathrm{Hom}_R(V, R)$ induced by B is an isomorphism), and with a quadratic mapping $\phi: V\to R$ such that $B(x, y) =\frac{1}{2}(\phi(x+y)-\phi(x)-\phi(y))$ and $\phi(rx)=r^2\phi(x)$ for all x, y in V and r in R. We denote the group of multiplicative units of R by $U(R)$. If (V, B, ϕ) is a free rank n quadratic space over R with an orthogonal basis $\{x_1, \cdots, x_n\}$, we will write $\langle a_1, \cdots, a_n\rangle$ for (V, B, ϕ) where the $a_i=\phi(x_i)$ are in $U(R)$, and denote the space by the table $[a_{ij}]$ where $a_{ij}=B(x_i, x_j)$. In the case $n=2$ and $B(x_1, x_2)=\frac{1}{2}$ we reserve the notation $[a_{11}, a_{22}]$ for the space.

A commutative ring R having 2 a unit is called full [10] if for every triple a_1, a_2, a_3 of elements in R with $(a_1, a_2, a_3)=R$, there is an element w in R such that $a_1+a_2w+a_3w^2=$unit.

LEMMA 1.1. *Let (V, B, ϕ) be a free quadratic space of rank n over a full ring R. Then,*

(i) *$(V, B, \phi)\simeq\langle a_1, \cdots, a_n\rangle$ where each a_i is in $U(R)$.*

(ii) *if n is even, then,*

$$V\simeq\mathop{\perp}_{i=1}^{n/2}[b_i, c_i],$$

where b_i and $1-4b_ic_i$ are in $U(R)$ for each i.

Proof. (i) is just [10, Theorem 3.2, p. 541], (ii) follows by (i), and by a change of bases.

A^s denotes the graded subalgebra of a (Z_2-)graded R-algebrat A generated by

$$\{a \in A_0 \text{ or } A_1 \mid as = (-1)^{\deg a \cdot \deg s} sa \text{ for all homogeneous } s \text{ in } S\}$$

where S is a subset of A.

Let $Q_2(R)$ be the set of isomorphism classes of graded separable R-algebras which are projective R-modules of rank two. When any such algebra is considered without its grading it is a Galois extension of R with a cyclic Galois group of order two [11, Corollary 7.4, p. 487]. An abelian group multiplication $*$ is defined on $Q_2(R)$ as follows: let L_1 and L_2 be two algebras representing classes in $Q_2(R)$. Then if σ_1, σ_2 are their non-trivial automorphisms we define $L_1 * L_2$ to be the isomorphism class of the subalgebra of $L_1 \hat{\otimes} L_2$ fixed by $\sigma_1 \hat{\otimes} \sigma_2$[11, p. 488]. Furthermore, there is an exact sequence of abelian groups

$$0 \to Br(R) \to BW(R) \to Q_2(R) \to 0.$$

Here, the first map is the one that takes the class of an ungraded algebra to the class of that algebra concentrated in degree 0, and the second takes the class of a non-degenerated algebra A (i.e., A_0 and A_1 have positive rank) to $L(A)$, where $L(A) =$ class (A^{A_0}) [11, Theorem 7.10, p. 490].

2. The Clifford algebra

Let (V, B, ϕ) be a quadratic space over R, and $T(V)$ the tensor algebra of V, with the obvious grading. We denote by $J(V)$ the ideal of $T(V)$ generated by the homogeneous elements $x \otimes x - \phi(x)$ for all x in V. Then $\mathrm{Cliff}(V)$ is defined to be the graded quotient algebra $T(V)/J(V)$. $\mathrm{Cliff}(V)$ is actually a graded central seperable R-algebra.

LEMMA 2.1. *Let (V, B, ϕ) be a free quadratic space over R, with* $\mathrm{Cliff}(V) = C = C_0 \oplus C_1$. *Then,*

(i) *if $(V, B, \phi) = \langle a \rangle$, then* $(\mathrm{Cliff}(V))^{C_0} = \mathrm{Cliff}(V) = R \oplus Rx$ *with $x^2 = a$, where ihe grading is given by $C_0 = R$, $C_1 = Rx$.*

(ii) *if $(V, B, \phi) = \langle a_1, \cdots, a_n \rangle$ with respect to a basis $\{x_1, \cdots, x_n\}$, then as an ungraded algebra* $\mathrm{Cliff}(V)$ *is a free R-algebra of rank 2^n with a homogeneous basis $\{x_1^{r_1} \cdots x_n^{r_n} \mid r_i = 0, 1\}$ where $x_i^2 = a_i$ $x_i x_j = -x_j x_i$ when $i \neq j$. The degree of $x_1^{r_1} \cdots x_n^{r_n}$ is congruent to Σr_i modulo 2.*

(iii) *if $(V, B, \phi) = [a, b]$, then* $\mathrm{Cliff}(V)$ *is the free rank 4 R-algebra with a bas is $\{1, x, y, xy\}$ where $x^2 = a$, $y^2 = b$, $xy + yx = 1$, and the grading is given by $C_0 = R \oplus Rxy$, $C_1 = Rx \oplus Ry$.* $(\mathrm{Cliff}(V)^{C_0}$ *is C_0.*

Proof. See [9, Example 2.2, p. 517].

LEMMA 2.2. *Let (V, B, ϕ) be a free quadratic space of even rank over a full ring R. Then* $|\mathrm{Cliff}(V)|$ *is a central separable (ungraded) R-algebra where*

$|\mathrm{Cliff}(V)|$ *is the algebra* $\mathrm{Cliff}(V)$ *considered as an ungraded algebra.*

Proof. Let $C=\mathrm{Cliff}(V)=C_0 \oplus C_1$. By Lemma 1.1, (ii), we can write $V=\perp V_i$, where each V_i is free of rank 2. But by [3, Lemma 3.1, p.147] $\mathrm{Cliff}(V)=\hat{\otimes}\,\mathrm{Cliff}(V_i)$, hence $L(\mathrm{Cliff}(V))=*L(\mathrm{Cliff}(V_i))$ since L is a group homomorphism. By Lemma 2.1. (iii), each representing algebra in $L(\mathrm{Cliff}(V_i))$ is concentrated in degree 0, hence the product in $Q_2(R)$ is concentrated in degree 0. Thus we have $C^{C_0} \subseteq C_0$. Now, let x be in the center of C considered as an ungraded algebra. Then x is in C^{C_0}, hence is in C_0. But if x is in C_0 and commutes with all of C, it is in C^C. Then since C is central as a graded algebra, $C^C=R$ and x is in R. Therefore C is central in the ungraded sense. Finally, we always know $|\mathrm{Cliff}(V)|$ is separable [11, Corollary 5.13. p.482].

If (V, B, ϕ) is a free quadratic space over R, we write $\det(V)$ for the determinant of B and $D(V)$ for the class of $\det(V)$ in $U(R)/(U(R))^2$.

For each i in $\mathbf{Z}_2$ and $a \in U(R)$ we will write $R\{a\}_i$ for the graded algebra $R \oplus Rz$ with $z^2=a$ and deg $z=i$. From Lemma 2.1, (i) and (iii) (with a change of bases), we see $L(\mathrm{Cliff}(\langle a_1 \rangle))=\mathrm{class}(R\{a_1\}_1)$ and $L(\mathrm{Cliff}(\langle a_1, a_2 \rangle))=\mathrm{class}\ (R\{-a_1a_2\}_0)$, thus $R\{a\}_i$ always represents a class in $Q_2(R)$. Clearly, the automorphism σ with $\sigma(z)=-z$ is the unique non-trivial automorphism of $R\{a\}_i$.

LEMMA 2.3. *Let R be a full ring. Then,*

(i) *if* (V, B, ϕ) *is a free quadratic space of rank n over R, then* $L(\mathrm{Cliff}(V))$ *is represented by* $R\{(-1)^{n(n-1)/2}\det(V)\}_t$ *where t is congruent to n modulo* 2.

(ii) *if* (V, B, ϕ) *and* (V', B', ϕ') *are two free spaces of the same rank over R, then* $L(\mathrm{Cliff}(V)) = L(\mathrm{Cliff}(V'))$ *if and only if* $D(V)=D(V')$.

Proof. (i) If σ_1 and σ_2 are the unique non-trivial automorphisms of $R\{a_1\}_i$ and $R\{a_2\}_j$, it is not hard to show that the subalgebra of $(R\{a_1\}_i \hat{\otimes} R\{a_2\}_j)$ fixed by $\sigma_1 \otimes \sigma_2$ is isomorphic to $R\{(-1)^{ij}a_1a_2\}_{i+j}$. The result follows easily by induction using Lemma 1.1, (i) and Lemma 2.1, (i).

(ii) By part (i) it will suffice to show that if an R-algebra has two R-bases $\{1, t\}$ and $\{1, u\}$ with t^2 and u^2 in R, then $t=bu$ for some b in $U(R)$. But since $\{1, u\}$ is a basis, we can write $t=a+bu$ for some a and b in R. Then $t^2=(a^2+b^2u^2)+2abu$, hence $ab=0$. But then $-a^2\cdot 1+a\cdot t=0$, thus by the independence of 1 and t we obtain $a=0$ and $t=bu$. Furthermore, since $\{1, t\}$ is a basis b must be a unit in R.

If a, b are units in a full ring R, we define the quatenion algebra $(a, b)/R$ to be the (ungraded) free rank 4 R-algebra with basis $\{1, x, y, xy\}$ subject to $x^2=a$, $y^2=b$, $xy=-xy$. By Lemma 2.1, (ii) we see that $[(a, b)/R]=|\mathrm{Cliff}(\langle a, b \rangle)|$, and is central separable by Lemma 2.2.

LEMMA 2.4. *Let R be a full ring. If* $a, b, c \in U(R)$, *then* $\mathrm{Cliff}\ (\langle a, b, c, abc \rangle) = [(-ab, -ac)/R]$ *in* $BW(R)$, *and are equal as ungraded algebras in* $Br(R)$.

Proof. See [9, Lemma 2.9, p. 521].

Let (V, B, ϕ) be a free quadratic space of rank n over a full ring R. We define $H(V)$ as the class of $\mathrm{Cliff}(V \perp n\langle -1\rangle \perp \langle 1, -\det(V)\rangle)$ in $BW(R)$. Then $H(\langle a\rangle) = \text{class}\ (\mathrm{Cliff}(\langle a, -1, 1, -a\rangle)) = 1$ since $\langle a, -1, 1, -a\rangle$ is hyperbolic, and if (V', B', ϕ') is another free quadratic space of rank m then, by Lemma 2.4, and [3, Theorem 3.9 and Corollary 3.10, p. 154],

$$\begin{aligned} &H(V)\cdot H(V')\cdot (H(V\perp V'))^{-1} \\ &\quad = \text{class}(\mathrm{Cliff}(\langle 1, -\det(V),\ -\det(V'),\ \det(V)\cdot\det(V')\rangle)) \\ &\quad = \text{class}([(\det(V),\ \det(V'))/R]) \text{ in } BW(R). \end{aligned}$$

Using these two observations and the bilinearity of the quaternion algebra in $Br(R)$, we easily obtain by induction that

$$H(\langle a_1, \cdots, a_n\rangle) = \text{class}(\prod_{i<j}[(a_i, a_j)/R]) \text{ in } BW(R).$$

And, by definition of $H(V)$, $H(V)$ is independent of the diagonalization of V.

If $(V, B, \phi) = \langle b_1, \cdots, b_n\rangle$ is a free quadratic space over a full fing R, we define the Hasse invariant, $\mathrm{Hasse}(V) = \text{class}(\prod_{i<j}[(b_i, b_j)/R])$ in $Br(R)$. Since $H(V)$ is concentrated in degree 0, $\mathrm{Hasse}(V) = H(V)$ in $BW(R)$.

THEOREM 2.5. *Let (V, B, ϕ) and (V', B', ϕ') be two free quadratic spaces of same rank over a full ring R. Then* $\mathrm{Cliff}(V) = \mathrm{Cliff}(V')$ *in $BW(R)$ if and only if* $\mathrm{Hasse}(V) = \mathrm{Hasse}(V')$ *and* $D(V) = D(V')$.

Proof. By Lemma 2.3, (ii), $\mathrm{Cliff}(V) = \mathrm{Cliff}(V')$ in $BW(R)$ implies $D(V) = D(V')$, thus by the definition of $H(V)$, $\mathrm{Cliff}(V) = \mathrm{Cliff}(V')$ in $BW(R)$ if and only if $H(V) = H(V')$ in $BW(R)$ and $D(V) = D(V')$. But since the map $Br(R) \to BW(R)$ is injective, $\mathrm{Hasse}(V) = \mathrm{Hasse}(V')$ if and only if $H(V) = H(V')$. This completes the proof.

References

1. C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik* 2, J. Reine Angew. Math. **183** (1941), 148-167.
2. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409.
3. H. Bass, *Lectures on topics in algebraic K-theory*, Tata Institute, 1967.
4. L. N. Childs, G. Garfinkel, and M. Orzech, *The Brauer group of graded Azumaya algebras*, Trans. Amer. Math. Soc. **175** (1973), 299-326.
5. B. Kirkwood and B.R. McDonald, *The Witt ring of a full ring*, J. Algebra **64** (1980), 148-166.
6. M. Knebusch and M. Kolster, *Wittrings*, Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1982.

7. ________, A. Rosenberg, and R. Ware, *Structure of Witt rings and quotients of abelian group rings*, Amer. J. Math. **94**(1972), 119–155.
8. T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin, Reading, Mass., 1973.
9. K. I. Mandelberg, *A note on quadratic forms over arbitrary semi-local rings*, Can. J. Math. **27**(1975), 513–527.
10. B. R. McDonald and B. Hershberger (Kirkwood), *The orthogonal group over a full ring*, J. Algebra **51**(1978), 536–549.
11. C. Small, *The Brauer-Wall group of a commutative ring*, Trans. Amer. Math. Soc. **156**(1971), 455–491.
12. E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. **176**(1937), 31–44.

Korea University
Seoul 132, Korea