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ON SUPER CONTINUOUS FUNCTIONS

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1. Introduction

B.M. Munshi and D.S. Bassan defined and developed the concept of super continuity in [5]. The concept has been investigated further by I.L. Reilly and M.K. Vamanamurthy in [6] where super continuity is characterized in terms of the semi-regularization topology. Super continuity is related to the concepts of δ-continuity and strong θ-continuity developed by T. Noiri in [7]. The purpose of this note is to derive relationships between super continuity and other strong continuity conditions and to develop additional properties of super continuous functions. Super continuity implies continuity, but the converse implication is false [5]. Super continuity is strictly between strong θ-continuity and δ-continuity and strictly between complete continuity and δ-continuity.

The symbols X and Y will denote topological spaces with no separation axioms assumed unless explicity stated. The closure and interior of a subset U of a space X will be denoted by $Cl(U)$ and $\text{Int}(U)$ respectively and U is said to be regular open (resp. regular closed) if $U=\text{Int}[Cl(U)$ (resp. $U=Cl(\text{Int}(U)]$. If necessary, a subscript will be added to denote the space in which the closure or interior is taken.

2. Relationships between super continuity and other strong continuities

DEFINITION 1. A function $f: X\to Y$ is said to be super continuous [5] (resp. δ-continuous [7], strongly θ-continuous [4]) if for each $x\in X$ and each open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f[\text{Int}(Cl(U))]\subset V$[resp. $f[\text{Int}(Cl(U))]\subset\text{Int}[Cl(V)]$, $f[Cl(U)]\subset V$]

DEFINITION 2. [2]. A function $f: X\to Y$ is said to be strongly continuous if for every subset A of X, $f[Cl(A)]\subset f(A)$

DEFINITION 3. [1]. A function $f: X\to Y$ is said to be completely continuous if for each open subset V of $Y, f^{-1}(V)$ is a regular open subset of X.

To begin with we collect and extend known relationships among various strong continuities. The following result is fairly obvious, but to my knowledge has not appeared in the literature.

THEOREM 1. *If* $f: X \to Y$ *is strongly continuous, then* f *is strongly* θ*-continuous.*

Proof. Let $x \in X$ and let V be an open neighborhood of $f(x)$ in Y. Let $U = f^{-1}(V)$. Since f is strongly continuous, f must be continuous. Hence U is open. Then since f is strongly continuous, $f[Cl(U)] = f[Cl(f^{-1}(V))] \subset f[f^{-1}(V)] \subset V$. Thus f is strongly θ-continuous.

From [1] strong continuity implies complete continuity and the converse implication does not hold.

From [5] a function is super continuous if and only if the inverse image of every open set is δ-open, that is the union of regularly open sets. Hence complete continuity implies super continuity.

Based on the above definitions and remarks the following implications are clear:

$$\text{strong continuity} \to \left| \begin{array}{l} \to \text{complete continuity} \to \\ \\ \to \text{strong } \theta\text{-continuity} \to \end{array} \right| \to \text{super continuity} \to \left| \begin{array}{l} \to \text{continuity} \\ \qquad\qquad (*) \\ \to \delta\text{-continuity} \end{array} \right.$$

The following example and remarks show that none of the converse implications in $(*)$ hold and that complete continuity and strong θ-continuity are independent.

Since δ-continuity and continuity are independent [7], neither δ-continuity nor continuity implies super continuity. The following example shows that super continuity does not imply either strong θ-continuity or complete continuity.

EXAMPLE 1. Let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$. Let $f: (X, T) \to (X, T)$ be the identity mapping. Then f is super continuous, but f is not strongly θ-continuous at $x = a$. Also f is not completely continuous because $\{a, c\}$ is open but not regular open.

Since R (the real number) with the usual topology is regular, the identity mapping on R is strongly θ-continuous. However, the identity mapping is clearly not strongly continuous. Also since R contains open sets that are not regular open, the identity mapping is not completely continuous. Hence strong θ-continuity does not imply either strong continuity or complete continuity.

In Example 6.1 of [1] Arya and Gupta give a completely continuous function that is not strongly continuous. It is easily seen that this function is also not strongly θ-continuous.

Hence none of the converse implications in $(*)$ hold and strong θ-continuity and complete continuity are independent. Conditions required for reversing some of the implications in $(*)$ will now be investigated.

DEFINITION 4. [5]. A space X is said to be semi-regular if for each x in X and each open neighborhood V of x there is an open neighborhood U of x such that $x \in U \subset \text{Int}[Cl(U)] \subset V$.

THEOREM 2. *If $f: X\to Y$ is δ-continuous and Y is semi-regular, then f is super continuous.*

Proof. Let $x\in X$ and let V be an open neighborhood of $f(x)$ in Y. Since Y is semi-regular, there exists an open neighborhood V_1 of $f(x)$ in Y such that $f(x) \in V_1\subset \mathrm{Int}[Cl(V_1)]\subset V$. Because f is δ-continuous there exists an open neighborhood U of x in X such that $f[\mathrm{Int}(Cl(U))]\subset \mathrm{Int}[Cl(V_1)]\subset V$. Thus f is super continuous.

This result slightly extends a result of *T.* Noiri's (Theorem 4.6 in [7]) in which it is stated that if $f: X\to Y$ is δ-continuous and Y is semi-regular, then f is continuous. From [5], if $f: X\to Y$ is continuous and X is semi-regular, then f is super continuous. Hence if X and Y are both semi-regular, then continuity, super continuity and δ-continuity are all equivalent.

DEFINITION 5. [7]. A space X is said to be almost-regular if for each regular closed subset F of X and each x in X-F, there exist disjoint open sets U and V in X such that $x\in U$ and $F\subset V$.

THEOREM 3. *If $f: X\to Y$ is super continuous and X is almost-regular, then f is strongly θ-continuous.*

Proof. Let $x\in X$ and let V be an open neighborhood of $f(x)$ in Y. Because f is super continuous, there is an open neighborhood U of x in X such that $f[\mathrm{Int}(Cl(U))]\subset V$. Then since Int $[Cl(U)]$ is regular open and X is almost-regular, there is a regular open set W for which $x\in W\subset Cl(W)\subset \mathrm{Int}[Cl(U)]$. Then obviously $f[Cl(W)]\subset f[\mathrm{Int}(Cl(U)]\subset V$. Hence f is strongly θ-continuous.

COROLLARY. *If $f: X\to Y$ is a function with X almost-regular and Y semi-regular, then strong θ-continuity, super continuity, and δ-continuity are all equivalent.*

3. Properties of super continuous functions

The next result follows from the straightforward use of definitions or from Theorem 1 in [6].

THEOREM 4. *Let $f: X\to Y$ be a function and let $g: X\to X\times Y$, given by $g(x) = [x, f(x)]$, be its graph function. Then g is super continuous if and only if f is super continuous and X is semi-regular.*

DEFINITION 5. [8]. A set U is said to be δ-open if for each x in U there is a regular open set V such that $x\in V\subset U$. A set is δ-closed if its complement is δ-open.

The following result is implied by the fact that the inverse image of an open set under a super continuous function is δ-open [5]. The proof is omitted.

THEOREM 5. *If $f, g : X \to Y$ are super continuous functions and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is δ-closed.*

THEOREM 6. *If $f : X \to Y$ is super continuous and A is an open subset of X, then $f|A : A \to Y$ is super continuous.*

Proof. Let $x \in A$ and let V be an open neighborhood of $f(x)$. Then there exists an open subset U of X such that $x \in U$ and $f\,[\mathrm{Int}_X(Cl_X(U))] \subset V$. Then

$$\begin{aligned} f[\mathrm{Int}_A(Cl_A(U \cap A))] &= f[A \cap \mathrm{Int}_X(A \cap Cl_X(U \cap A))] \\ &= f(A \cap \mathrm{Int}_X(A) \cap \mathrm{Int}_X(Cl_X(U \cap A))] \\ &= f[A \cap \mathrm{Int}_X(Cl_X(U \cap A))] \subset f[\mathrm{Int}_X(Cl_X(U))] \subset V. \end{aligned}$$

Hence $f|_A : A \to Y$ is super continuous.

The function in Example 1 is super continuous, but the restriction to the set $\{b, c\}$ is not super continuous. Thus some condition on the set A is necessary in order for $f|_A : A \to Y$ to be super continuous.

The proof of the next theorem follows easily from the definitions.

THEOREM 7. *If $f : X \to Y$ is δ-continuous and $g : Y \to Z$ is super continuous, then $g \circ f : X \to Z$ is super continuous.*

COROLLARY. *If $f : X \to Y$ and $g : Y \to Z$ are super continuous, then $g \circ f : X \to Z$ is super continuous.*

4. Sufficient conditions for super contunuity.

A condition that is weaker than δ-continuity is related to super continuity. We make the following definition.

DEFINITION 6. A function $f : X \to Y$ is said to weakly δ-continuous if for each $x \in X$ and each open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f[\mathrm{Int}(Cl(U))] \subset Cl(V)$.

The graph of a function $f : X \to Y$, denoted by $G(f)$, is the subset $\{[x, f(x)] : x \in X\}$ of the product space $X \times Y$. We make the following definition which is analogous to the definition of θ-closed with respect to X in [4].

DEFINTION 7. The graph of a function $f : X \to Y$ is said to be δ-closed with respect to X if for each $(x, y) \in X \times Y - G(f)$, there exist open sets U and V such that $x \in U \subset X$ and $y \in V \subset Y$ and $[\mathrm{Int}(Cl(U)] \times V) \cap G(f) = \phi$.

DEFINITION 8. [3]. A space Y is rim-compact if for every y in Y and every open neighborhood V of y, there is an open set W such that $y \in W \subset V$ and the boundary of $W[Bd(W)]$ is compact.

In the following sequence of results Theorems 8 and 9 and their proofs are analogous to results obtained for strongly θ-continuous functions by Long and Herrington. (Theorem 10 and 12 in [4]).

THEOREM 8. *If $f: X \to Y$ is weakly δ-continuous; Y is rim-compact, and $G(f)$ is δ-closed with respect to X, then f is super continuous.*

Proof. Let $x \in X$ and let V be an open neighborhood of $f(x)$. Since Y is rim-compact, there is an open set V_0 such that $f(x) \in V_0 \subset V$ and $Bd(V_0)$ is compact. Because f is weakly δ-continuous, there is an open neighborhood U of x in X such that $f[\mathrm{Int}(Cl(U))] \subset Cl(V_0)$. Let $y \in Bd(V_0)$. Since $f(x) \in V_0$ which is disjoint from $Bd(V_0)$, $(x, y) \notin G(f)$. Then since $G(f)$ is δ-closed with respect to X, there exist open sets A_y and B_y such that $x \in A_y$ and $y \in B_y$ and $f[\mathrm{Int}(Cl(A_y))] \cap B_y = \phi$. The collection $\{B_y : y \in Bd(V_0)\}$ is an open cover of $Bd(V_0)$ which is compact. Hence there is a finite collection $\{B_{y_1}, B_{y_2}, \cdots, B_{y_n}\}$ for which $Bd(V_0) \subset \bigcup_{i=1}^{n} B_{y_i}$. Let $U_0 = U \cap (\bigcap_{i=1}^{n} A_{y_i})$. Then

$$f[\mathrm{Int}(Cl(U_0))] \subset f[\mathrm{Int}(Cl(\bigcap_{i=1}^{n} A_{y_i}))] \subset f[\bigcap_{i=1}^{n} \mathrm{Int}(Cl(A_{y_i}))]$$
$$\subset \bigcap_{i=1}^{n} f(\mathrm{Int}(Cl(A_{y_i})))$$

which is disjoint from $\bigcup_{i=1}^{n} B_{y_i}$ and hence disjoint from $Bd(V_0)$. Thus $f[\mathrm{Int}(ClU_0))] \cap Bd(V_0) = \phi$. However $f[\mathrm{Int}(Cl(U_0))] \subset f[\mathrm{Int}(Cl(U))] \subset Cl(V_0)$. Therefore $f[\mathrm{Int}(Cl(U_0))] \subset Cl(V_0) - Bd(V_0) \subset V_0$. Hence f is super continuous.

COROLLARY. *If $f: X \to Y$ is δ-continuous; Y is rim-compact and $G(f)$ is δ-closed with respect to X, then f is super continuous.*

THEOREM 9. *If $f: X \to Y$ is weakly δ-continuous and Y is Hausdorff, then $G(f)$ is δ-closed with respect to X.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W such that $y \in W$ and $f(x) \in V$. Because f is weakly δ-continuos, there is an open neighborhood U of x for which $f[\mathrm{Int}(Cl(U))] \subset Cl(V)$. Then $(x, y) \in [\mathrm{Int}(Cl(U))] \times W$ and $f[\mathrm{Int}(Cl(U))] \subset Cl(V)$ which is disjoint from W. Therefore $[(\mathrm{Int}(Cl(U))] \times W] \cap G(f) = \phi$. Thus $G(f)$ is δ-closed with respect to X.

COROLLARY. *If $f: X \to Y$ is weakly δ-continuous, and Y is Hausdorff and rim-compact, then f is super continuous.*

The next theorem follows from the fact that $f: X \to Y$ is super continuous if and only if f is continuous with respect to the semi-regularization topology on X (Theorem 1, [6]) and that a function with a compact range space and a closed graph is continuous.

THEOREM 10. *If $f: X \to Y$ is a function with $G(f)$ δ-closed with respect to X and Y compact, then f is super continuous.*

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