ON THE RADIUS PROBLEM OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction

Let $A$ denote the family of functions $f$ which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$ and normalized such that $f(0) = 0 = f'(0) - 1$. Ruscheweyh [4] introduced the classes $\{K_n\}$ of functions $f \in A$ which satisfy the condition

\[ \text{Re}\{D^{n+1}f/D^nf\} > \frac{1}{2}, \quad z \in \Delta, \]

where

\[ D^nf = z(z^{-1}f)^{(n)}/n!, \quad n \in N_0 = \{0, 1, 2, \ldots\}. \]

He proved that $K_{n+1} \subset K_n$ for each $n \in N_0$.

Let $\{R_n(\alpha)\}$ denote the classes of functions $f \in A$ which satisfy the condition

\[ \text{Re}\{z(D^nf)'/D^nf\} > \alpha, \quad z \in \Delta \]

for some $\alpha (0 \leq \alpha < 1)$. We have $R_0(\alpha) \equiv S^*(\alpha)$ and $R_1(\alpha) \equiv K(\alpha)$ for $0 \leq \alpha < 1$, where $S^*(\alpha)$ and $K(\alpha)$ are the well known classes of functions of order $\alpha$ and convex of order $\alpha$, respectively. The classes $R_n \equiv R_n(0)$ were considered by Singh and Singh [6]. It is easy to see that for each $n \geq 0$, $R_n(\alpha) \subset R_n(0)$, and for each $n \geq 1$, $R_n(\alpha) \subset K_n$.

We first prove that $R_{n+1}(\alpha) \subset R_n(\alpha)$, $0 \leq \alpha < 1$, $n \in N_0$. These inclusion relations establish that $R_n(\alpha) \subset S^*(\alpha)$ for each $n \geq 0$ and $R_n(\alpha) \subset K(\alpha)$ for each $n \geq 1$. We then deal with the problem of the radius of $R_{n+1}(\alpha)$ in $R_n(\alpha)$.

2. Radius problem

We need the following Lemma due to I.S. Jack [3] which is also due to Suffridge [7].

**Lemma 1.** Let $w$ be a nonconstant and analytic function in $|z| < r < 1$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at $z_0$, then $z_0w'(z_0) = kw(z_0)$, where $k$ is a real number and $k \geq 1$.

**Lemma 2.** $R_{n+1}(\alpha) \subset R_n(\alpha)$ holds for all $n \in N_0$ and $\alpha (0 \leq \alpha < 1)$.
Proof. Let \( f \in R_{n+1}(\alpha) \). Define \( w \), analytic in \( \Delta \) by
\[
(2.1) \quad \frac{z(D^n f)' - \alpha (D^n f)}{D^n f} = 1 + \frac{(2\alpha - 1)w(z)}{1 + w(z)}.
\]
It is easy to see that \( w(0) = 0 \) and \( w(z) \neq -1 \) for \( z \in \Delta \). It suffices to show that \( |w(z)| < 1 \), \( z \in \Delta \).

Using the identity
\[
(2.2) \quad z(D^n f)' = (n+1) D^{n+1} f - nD^n f,
\]
we can rewrite (2.1) as
\[
(2.3) \quad \frac{D^{n+1} f}{D^n f} = \frac{(n+1) + (n+2\alpha - 1)w(z)}{(n+1)(1+w(z))}.
\]
Taking the logarithmic derivative of (2.3) we get
\[
(2.4) \quad \frac{z(D^{n+1} f)'}{D^{n+1} f} = 1 + \frac{(2\alpha - 1)w(z)}{1 + w(z)} - \frac{2(1 - \alpha)zw'(z)}{(1+w(z))(n+1 + (n+2\alpha - 1)w(z))}.
\]
We now claim that \( |w(z)| < 1 \) for all \( z \in \Delta \). For otherwise, by Lemma 1 there exists a point \( z_0 \in \Delta \) such that \( z_0w'(z_0) = kw(z_0) \) with \( |w(z_0)| = 1 \) and \( k \geq 1 \). Applying this result to (2.4) we obtain
\[
\text{Re} \left\{ \frac{z_0(D^{n+1} f(z_0)')}{D^{n+1} f(z_0)} \right\} \leq \alpha - \frac{k(1-\alpha)}{n+\alpha} \leq \alpha, \text{ for each } n \geq 0.
\]
This contradicts the hypothesis that \( f \in R_{n+1}(\alpha) \). Hence we conclude that \( |w(z)| < 1 \) for all \( z \in \Delta \). This completes the proof of Lemma.

In [1], Al-Amirí has obtained the radius of \( K_{n+1} \) in \( K_n \). In view of Lemma 2, we raise the natural question of finding the largest disk \( \Delta_r = \{ z : |z| < r, 0 < r \leq 1 \} \) so that if \( f \in R_n(\alpha) \), then
\[
\text{Re} \left\{ \frac{z(D^m f(z)')}{D^m f(z)} \right\} > \alpha, \ m > n, \ z \in \Delta_r.
\]
Let \( B \) denote the class of functions \( w(z) \) that are analytic in \( \Delta \) and satisfy the conditions (i) \( w(0) = 0 \) and (ii) \( |w(z)| < 1 \) for \( z \in \Delta \). We need the following Lemma due to Singh and Goel [5].

Lemma 3. Let \( P(z) = (1 + lw(z)) / (1 + w(z)) \), \( a = (1-lr^2) / (1-r^2) \), \( d = (1-l) r / (1-r^2) \), then for \( |z| = r, 0 \leq r < 1 \), we have
\[
\text{Re} \left( kP(z) + l/P(z) \right) \geq \frac{r^2|P(z) - l|^2 - |1 - P(z)|^2}{(1-r^2)|P(z)|} \geq \begin{cases} \frac{2[(1+k)(1+l)a]^{1/2} - 2a}{R_0 \geq R_1} \\
\frac{k+l + 2l(1+k)r + l(1+lk)r^2}{(1+r)(1+lr)}, \ R_0 \leq R_1 \end{cases}
\]
where \( R_0^2 = (1 + l)a/(1 + k) \), \( R_1 = a - d \), \( k \geq 1 \), \(-1 \leq l < 1\).

We now prove our main result in the following

**Theorem.** Suppose \( \alpha_0(n) \) is the smallest positive root of the equation

\[
(2.5) \quad 4(n^2 + 2n + 5)x^4 + 4(n^3 - n^2 + n - 13)x^3 - (12n^3 + 9n^2 \\
+ 58n - 15)x^2 + 4(2n^3 + 7n + 3)x - 4 = 0
\]

lying in the interval \((\alpha_1, \alpha_2)\), where \( \alpha_1 = (n - 1)^2/(n^2 + 2n + 5) \), \( \alpha_2 = 2/[2n + 1 + (4n^2 + 4n + 9)^{1/2}] \).

If \( f \in R_n(\alpha) \), then

\[
(2.6) \quad \text{Re}\left\{ \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \right\} > 0
\]

holds for

\[
(2.7) \quad |z| < r(n, \alpha) = \begin{cases} \quad r_1, & 0 \leq \alpha \leq \alpha_0(n) \\
\quad r_2, & \alpha_0(n) \leq \alpha < 1,
\end{cases}
\]

where

\[
(2.8) \quad r_1 = \frac{n + 1}{2 - 3a - \alpha n + \{(1 - \alpha)(3 - 5a - 2\alpha n + (1 - \alpha)n^2)\}^{1/2}}
\]

and

\[
(2.9) \quad r_2 = \left(\frac{\alpha(n^2 + 2n + 5) - (n - 1)^2}{4\alpha(n + \alpha) - (1 - \alpha)(n^2 - 1) + \{2\alpha (1 - \alpha)(-5n^2 + 8(2 - \alpha)(n + \alpha)\}^{1/2}} \right)^{1/2}.
\]

The bounds for \(|z|\) in (2.7) are sharp.

**Proof.** Since \( f \in R_n(\alpha) \), we have

\[
(2.10) \quad \frac{z(D^nf(z))'}{D^n f(z)} = \frac{1 + (2\alpha - 1)w(z)}{1 + w(z)},
\]

where \( w \in B \) for all \( z \) in \( A \). Using (2.2) in (2.10), we get

\[
(2.11) \quad \frac{D^{n+1}f(z)}{D^n f(z)} = \frac{n + 1 + (n + 2\alpha - 1)w(z)}{(n + 1)(1 + w(z))}.
\]

Taking logarithmic derivative of (2.11) we have

\[
(2.12) \quad \frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} = -n + (n + 1)P(z) - \frac{(1 - l)zw'(z)}{(1 + w(z))(1 + lw(z))},
\]

where \( P(z) = (1 + lw(z))/(1 + w(z)) \) and

\( l = (n + 2\alpha - 1)/(n + 1) \). An application of Dieudonné's Lemma [2] that

\[ |zw'(z) - w'(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}, \quad w \in B, \quad z \in A \]
to the second term of (2.12) yields

\begin{align}
(2.13) \quad \text{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} &\geq \frac{- (1+l+n(1-l))}{1-l} + \\
&+ \frac{1}{1-l} \left[\text{Re}\left\{((n+1) (1-l) + 1)P(z) + l/P(z)\right\}ight. \\
&\left. - \frac{r^2 |P(z) - l|^2 - |1 - P(z)|^2}{1-r^2 |P(z)|}\right],
\end{align}

We note that $-1 \leq l < 1$ for all $\alpha, n \ (0 \leq \alpha < 1, \ n \geq 0)$. Since $k = (n+1) (1-l) + 1 \geq 1$, on using Lemma 3 in (2.13), we obtain

\begin{align}
(2.14) \quad \text{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} &\geq \frac{- (2a + (1+l) + n(1-l))}{1-l} + \\
&+ \frac{2}{1-l} \left\{(2 + (n+1) (1-l) (1+l) a)^{1/2}\right\}
\end{align}

for $R_0 \geq R_1$, and

\begin{align}
(2.15) \quad \text{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} &\geq \frac{(n+2\alpha-1) (2\alpha-1) r^2 + 2(\alpha n+3\alpha-2) r + (n+1)}{\left((n+1)(1+r)(1+(n+2\alpha-1)r)\right)}
\end{align}

for $R_0 \leq R$, where $a = (n+1 - (n+2\alpha-1) r^2) / (1-r^2)$,

\[ R_0 = \frac{1}{n+1} \left\{\frac{(n+\alpha) (n+1 - (n+2\alpha-1) r^2)}{(2-\alpha) (1-r^2)}\right\}^{1/2} \]

and

\[ R_1 = \frac{n+1 + (n+2\alpha-1) r}{(1+r)(n+1)}. \]

Now $\text{Re}\{z(D^{n+1}f(z))/D^{n+1}f(z)\} > 0$ yields the equations

\begin{align}
(2.16) \quad F_1(r) = (2\alpha-1) (n+2\alpha-1) r^2 + 2(\alpha n+3\alpha-2) r + n+1 = 0
\end{align}

for $R_0 \leq R_1$ and

\begin{align}
(2.17) \quad F_2(r) = &\left[8\alpha^2-3\alpha-1-2(1-3\alpha)n-(-1-\alpha)n^2\right]r^4 \\
&-2\left[4\alpha(n+\alpha) -(1-\alpha)(n^2-1)\right]r^2 + \alpha(n^2+2n+5) - (n-1)^2 = 0
\end{align}

for $R_0 \geq R_1$.

The two minima given by (2.14) and (2.15) become equal to each other for such a $\alpha \ (0 \leq \alpha < 1)$ and $n \ (n \geq 0)$ for which $R_0 = R_1$. This equation reduces to

\begin{align}
(2.18) \quad &\left(n+2\alpha-1\right) \left(n+2\alpha-2\right) r^3 + \left(n+2\alpha-1\right) \left(n+6-2\alpha\right) r^2 \\
&- \left(n+1\right) \left(n+4\alpha-6\right) r - \left(n+1\right) \left(n+2\right) = 0.
\end{align}

We note that $F_1(0) = n+1 > 0$, and $F_1(1) = 2(2\alpha^2+\alpha(2n+1)-1) < 0$ if $\alpha < \alpha_2 = 2/[2n+1+(4n^2+4n+9)^{1/2}]$. Hence $F_1(r)$ has a root in $(0, 1)$ if $\alpha < \alpha_2$. Its smallest root in $(0, 1)$ for $\alpha < \alpha_2$ is $r_1$. Similarly, $F_2(0) = \alpha(n^2+2n+5) - (n-1)^2 > 0$ if $\alpha > \alpha_1 = (n-1)^2/(n^2+2n+5)$, and $F_2(1) = -4(1-\alpha) < 0$. Thus we conclude that the
smallest root in $(0,1)$ of $F_2(r)$ is $r_2$ if $\alpha > \alpha_1$. The transition point \(r_2\) for the two cases may be obtained by eliminating \(r\) from (2.16) and (2.18) is the smallest positive root $\alpha_0(n)$ of the equation

$$4(n^2+2n+5)\alpha^4 + 4(n^2-n^2+n-13)\alpha^3 - (12n^3+9n^2+58n-15)\alpha^2 + 4(2n^2+7n+3)\alpha - 4 = 0,$$

where $\alpha_0(n)$ lies in the interval $(\alpha_1, \alpha_2)$. This completes the proof of the theorem.

The functions given by

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{n+1-(n+2\alpha-1)z}{(n+1)(1-z)}$$

and

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{n+\alpha}{n+1} + \frac{(1-\alpha)(1-z^2)}{(n+1)(1-2z \cos \theta + z^2)},$$

where $\cos \theta$ is the solution of

$$\frac{n+1-2(n+\alpha)r \cos \theta + (n+2\alpha-1)r^2}{1-2r \cos \theta + r^2} = \left\{ \frac{(n+\alpha)(n+1-(n+2\alpha-1)r^2)}{(2-\alpha)(1-r^2)} \right\}^{1/2},$$

show that the results in the above Theorem are sharp.

**Remark 1.** For $n=0$, Theorem gives the radii of convexity of $S^*(\alpha)$, the results were obtained earlier by Singh and Goel [5].

**Remark 2.** For $n=0$, $\alpha=1/2$ the above Theorem yields $r_1=1$ and $r_1=(2\sqrt{3}-3)^{1/2}$. Because of what we have mentioned in the Theorem, $r_1=1$ is impossible. Hence $r_2=(2\sqrt{3}-3)^{1/2}$ is the radius of convexity for the class $S^*(1/2)$.

**References**


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