Bull. Korean Math. Soc. **22**(1985), No. 1, pp. 31-36

ON THE RADIUS PROBLEM OF CERTAIN ANALYTIC FUNCTIONS

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1. Introduction

Let A denote the family of functions f which are analytic in the unit disk $\Delta=\{z : |z|<1\}$ and normalized such that $f(0)=0=f'(0)-1$. Ruscheweyh [4] introduced the classes $\{K_n\}$ of functions $f\in A$ which satisfy the condition

$$(1.1) \qquad Re\{D^{n+1}f/D^nf\}>\frac{1}{2},\ z\,\varepsilon\,\Delta,$$

where

$$D^nf=z(z^{n-1}f)^{(n)}/n!,\ n\in N_0=\{0,1,2,\cdots\}.$$

He proved that $K_{n+1}\subset K_n$ for each $n\in N_0$.

Let $\{R_n(\alpha)\}$ denote the classes of functions $f\in A$ which satisfy the condition

$$(1.2) \qquad \mathrm{Re}\{z(D^nf)'/D^nf\}>\alpha,\ z\in\Delta$$

for some $\alpha(0\leq\alpha<1)$. We have $R_0(\alpha)\equiv S^*(\alpha)$ and $R_1(\alpha)\equiv K(\alpha)$ for $0\leq\alpha<1$, where $S^*(\alpha)$ and $K(\alpha)$ are the well known classes of functions of order α and convex of order α, respectively. The classes $R_n\equiv R_n(0)$ were considered by Singh and Singh [6]. It is easy to see that for each $n\geq 0$, $R_n(\alpha)\subset R_n(0)$, and for each $n\geq 1$, $R_n(\alpha)\subset K_n$.

We first prove that $R_{n+1}(\alpha)\subset R_n(\alpha)$, $0\leq\alpha<1$, $n\in N_0$. These inclusion relations establish that $R_n(\alpha)\subset S^*(\alpha)$ for each $n\geq 0$ and $R_n(\alpha)\subset K(\alpha)$ for each $n\geq 1$. We then deal with the problem of the radius of $R_{n+1}(\alpha)$ in $R_n(\alpha)$.

2. Radius problem

We need the following Lemma due to I. S. Jack [3] which is also due to Suffridge [7].

LEMMA 1. *Let w be a nonconstant and analytic function in $|z|<r<1$, $w(0)=0$. If $|w|$ attains its maximum value on the circle $|z|=r$ at z_0, then $z_0w'(z_0)=kw(z_0)$, where k is a real number and $k\geq 1$.*

LEMMA 2. *$R_{n+1}(\alpha)\subset R_n(\alpha)$ holds for all $n\in N_0$ and α $(0\leq\alpha<1)$.*

Proof. Let $f\in R_{n+1}(\alpha)$. Define w, analytic in Δ by

(2.1) $$\frac{z(D^n f)'}{D^n f}=\frac{1+(2\alpha-1)w(z)}{1+w(z)}.$$

It is easy to see that $w(0)=0$ and $w(z)\neq -1$ for $z\in\Delta$. It sufficies to show that $|w(z)|<1$, $z\in\Delta$.

Using the identity

(2.2) $$z(D^n f)'=(n+1)D^{n+1}f-nD^n f,$$

we can rewrite (2.1) as

(2.3) $$\frac{D^{n+1}f}{D^n f}=\frac{(n+1)+(n+2\alpha-1)w(z)}{(n+1)(1+w(z))}.$$

Taking the logarithmic derivative of (2.3) we get

(2.4) $$\frac{z(D^{n+1}f)'}{D^{n+1}f}=\frac{1+(2\alpha-1)w(z)}{1+w(z)}-\frac{2(1-\alpha)zw'(z)}{(1+w(z))(n+1+(n+2\alpha-1)w(z))}.$$

We now claim that $|w(z)|<1$ for all $z\in\Delta$. For otherwise, by Lemma 1 there exists a point $z_0\in\Delta$ such that $z_0w'(z_0)=kw(z_0)$ with $|w(z_0)|=1$ and $k\geq 1$. Applying this result to (2.4) we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}f(z_0)}\right\}&\leq\alpha-\frac{k(1-\alpha)}{n+\alpha}\\&\leq\alpha,\ \text{for each } n\geq 0.\end{aligned}$$

This contradicts the hypothesis that $f\in R_{n+1}(\alpha)$. Hence we conclude that $|w(z)|<1$ for all $z\in\Delta$. This completes the proof of Lemma.

In [1], Al-Amiri has obtained the radius of K_{n+1} in K_n. In view of Lemma 2, we raise the natural question of finding the largest disk $\Delta_r=\{z:|z|<r,\ 0<r\leq 1\}$ so that if $f\in R_n(\alpha)$, then

$$\operatorname{Re}\left\{\frac{z(D^m f(z))'}{D^m f(z)}\right\}>\alpha,\quad m>n,\quad z\in\Delta_r.$$

Let B denote the class of functions $w(z)$ that are analytic in Δ and satisfy the conditions (i) $w(0)=0$ and (ii) $|w(z)|<1$ for $z\in\Delta$. We need the following Lemma due to Singh and Goel [5].

LEMMA 3. *Let $P(z)=(1+lw(z))/(1+w(z))$, $a=(1-lr^2)/(1-r^2)$, $d=(1-l)r/(1-r^2)$, then for $|z|=r$, $0\leq r<1$, we have*

$$\operatorname{Re}(kP(z)+l/P(z))-\frac{r^2|P(z)-l|^2-|1-P(z)|^2}{(1-r^2)|P(z)|}$$
$$\geq\begin{cases}2[(1+k)(1+l)a]^{1/2}-2a, & R_0\geq R_1\\ \dfrac{k+l+2l(1+k)r+l(1+lk)r^2}{(1+r)(1+lr)}, & R_0\leq R_1\end{cases}$$

where $R_0^2=(1+l)a/(1+k)$, $R_1=a-d$, $k\geq 1$, $-1\leq l<1$.

We now prove our main result in the following

THEOREM. *Suppose* $\alpha_0(n)$ *is the smallest positive root of the equation*

$$4(n^2+2n+5)x^4+4(n^3-n^2+n-13)x^3-(12n^3+9n^2+58n-15)x^2+4(2n^3+7n+3)x-4=0 \tag{2.5}$$

lying in the interval (α_1,α_2), *where* $\alpha_1=(n-1)^2/(n^2+2n+5)$, $\alpha_2=2/\{2n+1+(4n^2+4n+9)^{1/2}\}$.
If $f\in R_n(\alpha)$, *then*

$$\operatorname{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\}>0 \tag{2.6}$$

holds for

$$|z|<r(n,\alpha)=\begin{cases} r_1, & 0\leq\alpha\leq\alpha_0(n) \\ r_2, & \alpha_0(n)\leq\alpha<1, \end{cases} \tag{2.7}$$

where

$$r_1=\frac{n+1}{2-3\alpha-\alpha n+\{(1-\alpha)(3-5\alpha-2\alpha n+(1-\alpha)n^2)\}^{1/2}} \tag{2.8}$$

and

$$r_2=\left\{\frac{\alpha(n^2+2n+5)-(n-1)^2}{4\alpha(n+\alpha)-(1-\alpha)(n^2-1)+\{2\alpha(1-\alpha)(-5n^2+8(2-\alpha)(n+\alpha)\}^{1/2}}\right\}^{1/2}. \tag{2.9}$$

The bounds for $|z|$ *in* (2.7) *are sharp.*

Proof. Since $f\in R_n(\alpha)$, we have

$$\frac{z(D^nf(z))'}{D^nf(z)}=\frac{1+(2\alpha-1)w(z)}{1+w(z)}, \tag{2.10}$$

where $w\in B$ for all z in Δ. Using (2.2) in (2.10), we get

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{n+1+(n+2\alpha-1)w(z)}{(n+1)(1+w(z))}. \tag{2.11}$$

Taking logarithmic derivative of (2.11) we have

$$\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}=-n+(n+1)P(z)-\frac{(1-l)zw'(z)}{(1+w(z))(1+lw(z))} \tag{2.12}$$

where $P(z)=(1+lw(z))/(1+w(z))$ and
$l=(n+2\alpha-1)/(n+1)$. An application of Dieudonne's Lemma [2] that

$$|zw'(z)-w(z)|\leq\frac{|z|^2-|w(z)|^2}{1-|z|^2},\quad w\in B,\quad z\in\Delta$$

to the second term of (2.12) yields

$$\text{(2.13)} \qquad \operatorname{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} \geq \frac{-(1+l+n(1-l))}{1-l} + \\ +\frac{1}{1-l}\Big[\operatorname{Re}\{((n+1)(1-l)+1)P(z)+l/P(z)\} \\ -\frac{r^2|P(z)-l|^2-|1-P(z)|^2}{(1-r^2)|P(z)|}\Big].$$

We note that $-1\leq l<1$ for all $\alpha, n\,(0\leq\alpha<1,\ n\geq 0)$. Since $k=(n+1)(1-l)+1\geq 1$, on using Lemma 3 in (2.13), we obtain

$$\text{(2.14)} \qquad \operatorname{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} \geq \frac{-(2a+(1+l)+n(1-l))}{1-l} \\ +\frac{2}{1-l}\{(2+(n+1)(1-l))(1+l)a\}^{1/2}$$

for $R_0\geq R_1$, and

$$\text{(2.15)} \qquad \operatorname{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}\right\} \geq \frac{(n+2\alpha-1)(2\alpha-1)r^2+2(\alpha n+3\alpha-2)r+(n+1)}{(n+1)(1+r)(1+(n+2\alpha-1)r)}$$

for $R_0\leq R$, where $a=(n+1-(n+2\alpha-1)r^2)/(1-r^2)$,

$$R_0=\frac{1}{n+1}\left\{\frac{(n+\alpha)(n+1-(n+2\alpha-1)r^2)}{(2-\alpha)(1-r^2)}\right\}^{1/2}$$

and

$$R_1=\frac{n+1+(n+2\alpha-1)r}{(1+r)(n+1)}.$$

Now $\operatorname{Re}\{z(D^{n+1}f(z))'/D^{n+1}f(z)\}>0$ yields the equations

$$\text{(2.16)} \qquad F_1(r)=(2\alpha-1)(n+2\alpha-1)r^2+2(\alpha n+3\alpha-2)r+n+1=0$$

for $R_0\leq R_1$ and

$$\text{(2.17)} \qquad F_2(r)=[8\alpha^2-3\alpha-1-2(1-3\alpha)n-(1-\alpha)n^2]r^4 \\ -2[4\alpha(n+\alpha)-(1-\alpha)(n^2-1)]r^2+\alpha(n^2+2n+5)-(n-1)^2=0$$

for $R_0\geq R_1$.

The two minima given by (2.14) and (2.15) become equal to each other for such a $\alpha\,(0\leq\alpha<1)$ and $n\,(n\geq 0)$ for which $R_0=R_1$. This equation reduces to

$$\text{(2.18)} \qquad (n+2\alpha-1)(n+2\alpha-2)r^3+(n+2\alpha-1)(n+6-2\alpha)r^2 \\ -(n+1)(n+4\alpha-6)r-(n+1)(n+2)=0.$$

We note that $F_1(0)=n+1>0$, and $F_1(1)=2(2\alpha^2+\alpha(2n+1)-1)<0$ if $\alpha<\alpha_2=2/[2n+1+(4n^2+4n+9)^{1/2}]$. Hence $F_1(r)$ has a root in $(0,1)$ if $\alpha<\alpha_2$. Its smallest root in $(0,1)$ for $\alpha<\alpha_2$ is r_1. Similarly, $F_2(0)=\alpha(n^2+2n+5)-(n-1)^2>0$ if $\alpha>\alpha_1=(n-1)^2/(n^2+2n+5)$, and $F_2(1)=-4(1-\alpha)<0$. Thus we conclude that the smallest root in $(0,1)$ of $F_2(r)$ is r_2 if $\alpha>\alpha_1$. The transition point for the two cases may be obtained by eliminating r from (2.16) and (2.18) is the smallest positive root $\alpha_0(n)$ of the equation

$$4(n^2+2n+5)\alpha^4+4(n^3-n^2+n-13)\alpha^3-(12n^3+9n^2+58n-15)\alpha^2+4(2n^2+7n+3)\alpha-4=0,$$

where $\alpha_0(n)$ lies in the interval (α_1,α_2). This completes the proof of the theorem.

The functions given by

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{n+1-(n+2\alpha-1)z}{(n+1)(1-z)}$$

and

$$\frac{D^{n+1}f(z)}{D^nf(z)}=\frac{n+\alpha}{n+1}+\frac{(1-\alpha)(1-z^2)}{(n+1)(1-2z\cos\theta+z^2)},$$

where $\cos\theta$ is the solution of

$$\frac{n+1-2(n+\alpha)r\cos\theta+(n+2\alpha-1)r^2}{1-2r\cos\theta+r^2}=\left\{\frac{(n+\alpha)(n+1-(n+2\alpha-1)r^2}{(2-\alpha)(1-r^2)}\right\}^{1/2}$$

show that the results in the above Theorem are sharp.

REMARK 1. For $n=0$, Theorem gives the radii of convexity of $S^*(\alpha)$, the results were obtained earlier by Singh and Goel [5].

REMARK 2. For $n=0$, $\alpha=1/2$ the above Theorem yields $r_1=1$ and $r_1=(2\sqrt{3}-3)^{1/2}$. Because of what we have mentioned in the Theorem, $r_1=1$ is impossible. Hence $r_2=(2\sqrt{3}-3)^{1/2}$ is the radius of convexity for the class $S^*(1/2)$.

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