ON NEW CLASSES OF UNIVALENT FUNCTIONS
WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

\begin{equation}
    f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\end{equation}

which are analytic in the unit disk $\mathcal{U} = \{ z : |z| < 1 \}$. We denote by $\mathcal{S}$ the subclass of univalent functions $f(z)$ in $\mathcal{A}$ and by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{S}$ whose members are starlike with respect to the origin and convex in the unit disk $\mathcal{U}$, respectively. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in the unit disk $\mathcal{U}$ if and only if

\begin{equation}
    \text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha
\end{equation}

for $z \in \mathcal{U}$. Further, a function $f(z)$ in $\mathcal{A}$ is said to be convex of order $\alpha (0 \leq \alpha < 1)$ in the unit disk $\mathcal{U}$ if and only if

\begin{equation}
    \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha
\end{equation}

for $z \in \mathcal{U}$. We denote by $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively. Then, it is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}$ for $0 < \alpha < 1$ and that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$ for $\alpha = 0$.

Ruscheweyh [9] introduced the classes $\mathcal{K}_n$ of functions $f(z)$ in $\mathcal{A}$ satisfying

\begin{equation}
    \text{Re} \left\{ \frac{(z^n f(z))^{(n+1)}}{\left( z^{n-1} f(z) \right)^{(n)}} \right\} > \frac{n+1}{2}
\end{equation}

for $n \in \mathcal{N} \cup \{0\}$ ($\mathcal{N} = \{1, 2, 3, \cdots \}$) and $z \in \mathcal{U}$ and showed the basic property

\begin{equation}
    \mathcal{K}_{n+1} \subset \mathcal{K}_n
\end{equation}

for each $n \in \mathcal{N} \cup \{0\}$, where $\mathcal{K}_0 \equiv \mathcal{S}^*(1/2)$ and $\mathcal{K}_1 \equiv \mathcal{K}$.

Let

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\[ D^nf(z) = \frac{z^n f^n(z)}{n!} \]

for \( n \in \mathbb{N} \cup \{0\} \). This symbol \( D^nf(z) \) was named the \( n \)th order Ruscheweyh derivative of \( f(z) \) by Al-Amiri [2]. We note that \( D^0f(z) = f(z) \) and \( D^1f(z) = z f'(z) \).

The Hadamard product of two functions \( f(z) \in \mathcal{A} \) and \( g(z) \in \mathcal{A} \) will be denoted by \( f \ast g(z) \), that is, if \( f(z) \) is given by \( (1.1) \) and \( g(z) \) is given by \( (1.1) \) and \( g(z) \) given by

\[ g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \]

then

\[ f \ast g (z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \]

By using the Hadamard product, Ruscheweyh [9] defined that

\[ D^\beta f(z) = \left( \frac{z}{(1-z)^{\beta+1}} \right)^* f(z) \quad (\beta \geq -1) \]

which implies \( (1.6) \) for \( \beta \in \mathbb{N} \cup \{0\} \). With this notation \( (1.9) \), we can observe that the necessary and sufficient condition for a function \( f(z) \) in \( \mathcal{A} \) to be in the class \( \mathcal{K}_n \) is

\[ \text{Re} \left( \frac{D^{n+1}f(z)}{D^nf(z)} \right) > \frac{1}{2} \quad (z \in \mathbb{U}). \]

Furthermore, in this notation \( (1.10) \) also a class \( \mathcal{K}_{-1} \) can be defined as the class of functions \( f(z) \) in \( \mathcal{A} \) satisfying

\[ \text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{2} \quad (z \in \mathbb{U}). \]

Since \( \mathcal{K}_0 \equiv S^*(1/2) \subset S^* \subset S \), Ruscheweyh's result implies that \( \mathcal{K}_n \subset S^* \subset S \) for each \( n \in \mathbb{N} \cup \{0\} \).

Recently, by using the \( n \)th order Ruscheweyh derivative of \( f(z) \), Singh and Singh [10] introduced the subclass \( \mathcal{R}_n \) of \( \mathcal{A} \) whose members are characterized by the following condition

\[ \text{Re} \left( \frac{D^{n+1}f(z)}{D^nf(z)} \right) > \frac{n}{n+1} \quad (z \in \mathbb{U}) \]

for \( n \in \mathbb{N} \cup \{0\} \). We can see that \( \mathcal{R}_0 = \mathcal{C}^* \) and \( \mathcal{R}_n \subset \mathcal{K}_n \) for each \( n \in \mathbb{N} \), immediately. Hence \( \mathcal{R}_n \) is a subclass of \( \mathcal{C}^* \subset \mathcal{C} \) for each \( n \in \mathbb{N} \cup \{0\} \). Further we can show that \( \mathcal{R}_{n+1} \subset \mathcal{R}_n \) for every \( n \in \mathbb{N} \cup \{0\} \).

In the recent years, many classes of functions defined by using the \( n \)th order Ruscheweyh derivative of \( f(z) \) were studied by Al–Amiri [2], [3], Bulboaca [4],
Goel and Sohi [5], [6] and Owa [7], [8].

In the present paper we introduce the following classes $\mathcal{R}_n^*$ by using the $n$th order Ruscheweyh derivative of $f(z)$.

**Definition.** We say that $f(z)$ is in the class $\mathcal{R}_n^* \ (n \in \mathbb{N} \cup \{0\})$, if $f(z)$ defined by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0)$$

satisfies the condition (1.12) for $n \in \mathbb{N} \cup \{0\}$.

2. **Distortion theorems**

**Theorem 1.** Let the function $f(z)$ be defined by (1.13). Then $f(z)$ is in the class $\mathcal{R}_n^*$ if and only if

$$\sum_{k=2}^{\infty} \frac{(k+n-1)!(k)}{(k-1)!} a_k \leq n!.$$ 

Equality holds for the function defined by

$$f(z) = z - \frac{1}{2(n+1)} z^2.$$ 

**Proof.** Assume that the inequality (2.1) holds and let $|z| = 1$. Then we get

$$\left| \frac{D^{n+1}f(z)}{D^nf(z)} - 1 \right|$$

$$\leq \left| \frac{\sum_{k=2}^{\infty} (k+n-1)(k+n-2)\cdots k(k-1)a_k z^{k-1}}{(n+1)! - (n+1) \sum_{k=2}^{\infty} (k+n-1)(k+n-2)\cdots k a_k z^{k-1}} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} (k+n-1)(k+n-2)\cdots k(k-1)a_k}{(n+1)! - (n+1) \sum_{k=2}^{\infty} (k+n-1)(k+n-2)\cdots k a_k z^{k-1}}$$

$$\leq \frac{1}{n+1}.$$ 

This shows that the values of $D^{n+1}f(z)/D^nf(z)$ lie in a circle centered at $w=1$ whose radius is $1/(n+1)$. Thus we can observe that the function $f(z)$ satisfies (1.12) hence further, $f(z) \in \mathcal{R}_n^*$.

For the converse, assume that the function $f(z)$ belongs to the class $\mathcal{R}_n^*$. Then we have
for $z \in \mathbb{U}$. Choose values of $z$ on the real axis so that $D^{n+1}f(z)/D^nf(z)$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we get

\begin{equation}
(n+1)! - \sum_{k=2}^{\infty} (k+n) (k+n-1) \cdots ka_k z^{k-1}
\end{equation}

\begin{equation}
> \frac{n}{n+1}
\end{equation}

which gives (2.1).

Further we can see that the function $f(z)$ given by (2.2) is an extreme one for the theorem. Thus we have the theorem.

**Corollary 1.** Let the function $f(z)$ defined by (1.13) be in the class $\mathcal{R}_n^*$. Then

\begin{equation}
a_k \leq \frac{n!(k-1)!}{(k+n-1)!k}
\end{equation}

for $k \geq 2$. The equality holds for the function $f(z)$ of the form

\begin{equation}
f(z) = z - \frac{n!(k-1)!}{(k+n-1)!k} z^k.
\end{equation}

**Theorem 2.** Let the function $f(z)$ defined by (1.13) be in the class $\mathcal{R}_n^*$. Then we have

\begin{equation}
|f(z)| \geq |z| - \frac{1}{2(n+1)} |z|^2
\end{equation}

and

\begin{equation}
|f(z)| \leq |z| + \frac{1}{2(n+1)} |z|^2
\end{equation}

for $z \in \mathbb{U}$. The results are sharp.

**Proof.** Since $f(z) \in \mathcal{R}_n^*$, in view of Theorem 1, we obtain

\begin{equation}
(n+1)! 2 \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k
\end{equation}

\begin{equation}
\leq n!
\end{equation}
which implies that

\[(2.11) \quad \sum_{k=2}^{\infty} a_k \leq \frac{1}{2(n+1)}.\]

Therefore we can show that

\[(2.12) \quad |f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{1}{2(n+1)} |z|^2 \]

and

\[(2.13) \quad |f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{1}{2(n+1)} |z|^2 \]

for \(z \in \mathbb{U}\).

Finally, by taking the function

\[(2.14) \quad f(z) = z - \frac{1}{2(n+1)} z^2,\]

we can see that the results of the theorem are sharp.

**Corollary 2.** Let the function \(f(z)\) defined by (1.13) be in the class \(\mathcal{R}_n^*\). Then \(f(z)\) is included in a disk with its center at the origin and radius \(r\) given by

\[(2.15) \quad r = \frac{2n+3}{2(n+1)}.\]

**Theorem 3.** Let the function \(f(z)\) defined by (1.13) be in the class \(\mathcal{R}_n^*\). Then we have

\[(2.16) \quad |f''(z)| \geq 1 - \left(\frac{1}{n+1}\right) |z| \]

and

\[(2.17) \quad |f'(z)| \leq 1 + \left(\frac{1}{n+1}\right) |z| \]

for \(z \in \mathbb{U}\). The results are sharp.

**Proof.** In view of Theorem 1, we obtain

\[(2.18) \quad (n+1)! \sum_{k=2}^{\infty} ka_k \leq \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k \leq n!\]
which implies that

\[(2.19)\quad \sum_{k=2}^{\infty} k a_k \leq \frac{1}{n+1}.
\]

Hence, with the aid of (2.19), we have

\[
|f'(z)| \geq 1 - |z| \left( \sum_{k=2}^{\infty} k a_k \right)
\geq 1 - \left( \frac{1}{n+1} \right) |z|
\]

and

\[
|f'(z)| \leq 1 + |z| \left( \sum_{k=2}^{\infty} k a_k \right)
\leq 1 + \left( \frac{1}{n+1} \right) |z|
\]

for \(z \in U\). Further the results of the theorem are sharp for the function \(f(z)\) given by (2.14).

**Corollary 3.** Let the function \(f(z)\) defined by (1.13) be in the class \(\mathcal{R}_{n^*}\). Then \(f'(z)\) is included in a disk with its center at the origin and radius \(R\) given by

\[
(2.22) \quad R = \frac{n+2}{n+1}.
\]

**Theorem 4.** Let the function \(f(z)\) defined by (1.13) be in the class \(\mathcal{R}_{n^*}\). Then we have

\[
|f''(z)| < \frac{2}{n+1}
\]

for \(z \in U\).

**Proof.** With the aid of Theorem 1, we can see that

\[
(2.24) \quad \frac{(n+1)!}{2} \sum_{k=2}^{\infty} k(k-1) a_k \leq \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_k
\leq n!
\]

which gives that

\[
(2.25) \quad \sum_{k=2}^{\infty} k(k-1) a_k < \frac{2}{n+1}.
\]

Consequently we have

\[
|f''(z)| \leq \sum_{k=2}^{\infty} k(k-1) a_k |z|^{k-2}
\leq \frac{2}{n+1}.
\]
COROLLARY 4. Let the function \( f(z) \) defined by (1.13) be in the class \( \mathcal{R}_n^* \). Then \( f^n(z) \) is included in a disk with its center at the origin and radius \( 2(n+1) \).

3. Closure theorems

THEOREM 5. Let the functions

\[
 f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0)
\]

be in the class \( \mathcal{R}_n^* \) for every \( i = 1, 2, 3, \ldots, m \). Then the function \( h(z) \) defined by

\[
 h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0)
\]
is also in the same class \( \mathcal{R}_n^* \), where

\[
 \sum_{i=1}^{m} c_i = 1.
\]

Proof. By means of the definition of \( h(z) \), we obtain

\[
 h(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{m} c_i a_{k,i} \right) z^k.
\]

Further, since \( f_i(z) \) are in \( \mathcal{R}_n^* \) for every \( i = 1, 2, 3, \ldots, m \), we get

\[
 \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_{k,i} \leq n!
\]

for every \( i = 1, 2, 3, \ldots, m \). Hence we can see that

\[
 \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} \left( \sum_{i=1}^{m} c_i a_{k,i} \right) 
\]

\[
 = \sum_{i=1}^{m} c_i \left\{ \sum_{k=2}^{\infty} \frac{(k+n-1)!k}{(k-1)!} a_{k,i} \right\} 
\]

\[
 \leq \left( \sum_{i=1}^{m} c_i \right) n! 
\]

\[
 = n!
\]

with the aid of (3.5). This proves that the function \( h(z) \) is in the class \( \mathcal{R}_n^* \) by means of Theorem 1. Thus we have the theorem.

THEOREM 6. Let

\[
 f_1(z) = z
\]

and

\[
 f_k(z) = z - \frac{(k-1)!n!}{(k+n-1)!k} z^k
\]
for $k \in \mathbb{N} - \{1\}$. Then $f(z)$ is in the class $R_n^*$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ and

$$\sum_{k=1}^{\infty} \lambda_k = 1$$

**Proof.** Assume that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

$$= z - \sum_{k=2}^{\infty} \frac{(k-1)! n!}{(k+n-1)! k} \lambda_k z^k.$$

Then we have

$$\sum_{k=2}^{\infty} \left\{ \frac{(k+n-1)! k}{(k-1)!} \frac{(k-1)! n!}{(k+n-1)! k} \lambda_k \right\} \leq n!.$$

Consequently we can see that $f(z)$ is in the class $R_n^*$ by means of Theorem 1. Conversely, suppose that $f(z)$ is in the class $R_n^*$. Again, with the aid of Theorem 1, we get

$$a_k \leq \frac{(k-1)! n!}{(k+n-1)! k}$$

for $k \in \mathbb{N} - \{1\}$. Setting

$$\lambda_k = \frac{(k+n-1)! k}{(k-1)! n!} a_k$$

for $k \in \mathbb{N} - \{1\}$ and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k,$$

we obtain the representation (3.9). This completes the proof of the theorem.

**4. Modified Hadamard product**

Let $f(z)$ be defined by (1.13) and $g(z)$ be defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

($b_k \geq 0$).

And let $f \ast g(z)$ denote the modified Hadamard product of $f(z)$ and $g(z)$, that is,

$$f \ast g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$
THEOREM 7. Let the functions $f_i(z)$ defined by (3.1) be in the class $\mathcal{R}_{n_i}$ for each $i=1, 2, 3, \ldots, m$, respectively. Then the modified Hadamard product $f_1*f_2*\ldots*f_m(z)$ belongs to the class $\mathcal{R}_n^*$, where $n=\min_{1 \leq i \leq m} \{n_i\}$.

Proof. Since $f_i(z) \in \mathcal{R}_{n_i}^*$ for each $i=1, 2, 3, \ldots, m$, we have

$$a_{k,i} \leq \frac{1}{2(n_i+1)}$$

(i=1, 2, 3\ldots, m)

by using Theorem 1. Therefore we can see that

$$\sum_{k=2}^{\infty} \frac{(k+n-1)!k^m}{(k-1)!} \left( \prod_{i=1}^{m} a_{k,i} \right) \leq \frac{n!2(n+1)}{\prod_{i=1}^{m} 2(n_i+1)} \leq n!$$

with the aid of (4.3), where $n=\min_{1 \leq i \leq m} \{n_i\}$. This shows that the modified Hadamard product $f_1*f_2*\ldots*f_m(z)$ is in the class $\mathcal{R}_n^*$.

COROLLARY 5. Let the functions $f_i(z)$ defined by (3.1) be in the same class $\mathcal{R}_n^*$ for every $i=1, 2, 3, \ldots, m$. Then the modified Hadamard product $f_1*f_2*\ldots*f_m(z)$ also is in the class $\mathcal{R}_n^*$.

References


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