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SOME GENERALIZATION OF THE LANG'S EXISTENCE OF RATIONAL PLACE THEOREM

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1. Introduction

Let K be a real function field over a real closed field F. Then there exists an F-place $\varphi: K \to F \cup \{\infty\}$. This is Lang's Existence of Rational Place Theorem (6). There is an equivalent version of Lang's Theorem in (4). That is, if K is a function field over a field F, then, for any ordering P_0 on F which extends to K, there exists an F-place φ: $K \to F' \cup \{\infty\}$ where F' is a real closure of (F, P_0).

In [2], Knebusch pointed out the converse of the version of Lang's Theorem is also true.

By a valuation theoretic approach to Lang's Theorem, we have found out the following generalization of Lang and Knebusch's Theorem. Let K be an arbitrary extension field of a field F. Then an ordering P_0 on F can be extended to an ordering P on K iff there exists an F-place of K into some real closed field R containing F. Of course $R^2 \cap F = P_0$. The restriction K being a function field of F is vanished, though the codomain of the F-place is slightly varied. Therefore our theorem is a generalization of Lang and Knebusch's theorem.

2. Preliminaries and Main Theorem

By an ordering on a field F, we mean a subset $P \subsetneq F$ such that $P+P \subseteq P$, $P \cdot P \subseteq P$ and $P \cup (-P) = F$. From these axioms, it is easy to see that $P \cap (-P) = \{0\}$ [4].

The set of all orderings on F will be denoted by X_F. If $P \in X_F$, then the pair (F, P) is called an ordered field. For an extension field K of F, an ordering $Q \in X_K$ is said to extend $P \in X_F$, if $Q \cap F = P$. By a valuation on a field F, we shall always mean Krull valuation $v: F \to \Gamma$ onto an ordered group Γ, satisfying the two axioms

$$(1)\ v(xy) = v(x) + v(y) \text{ for any } x, y \in \dot{F} = F \setminus \{0\},$$
$$(2)\ v(x+y) \geqslant \min\{v(x),\ v(y)\} \text{ for } x, y,\ x+y \in \dot{F}.$$

For a given valuation v as above, we can define the following collection of associated objects.

$A := \{x \in F \mid x=0 \text{ or } v(x) \geqslant 0\}$ (the valuation ring of v),
$\mathfrak{M} := \{x \in F \mid x=0 \text{ or } v(x) > 0\}$ (the maximal ideal of v),
$U := A \setminus \mathfrak{M}$ (the group of valuation units),

$\bar{F}:=A/\mathfrak{M}$ (the residue class field of v),
$\pi: F \to \bar{F} \cup \{\infty\}$ denote the place associated with v,
defined by $\pi(x) = \begin{cases} x+\mathfrak{M} & \text{if } x \in A \\ \infty & \text{if } x \notin A. \end{cases}$

We usually write $\bar{x}$ for $x+\mathfrak{M}$, and say $(v, A, \mathfrak{M}, \Gamma, \cdots)$ is a valuation instead of saying v is a valuation. We shall write $a \geqslant_P b$ if $a-b \in P$, and $a >_P b$ if $a-b \in \dot{P} = P \setminus \{0\}$.

THEOREM 1. *Let $P \in X_F$, and $(v, A, \mathfrak{M}, \Gamma, \cdots)$ be a valuation on F. Then the following statements are equivalent* (5).

(1) $0 <_P a \leqslant_P b \Rightarrow v(a) \geqslant v(b)$ in Γ.
(2) *A is convex with respect to P.*
(3) *$\mathfrak{M}$ is convex with respect to P.*
(4) $1+\mathfrak{M} \subseteq P$.

DEFINITION 1. If any (and hence all) of the conditions in Th. 1 holds for v and P, we shall say v is *compatible with P* (*or that P is compatible with v*).

In case P is compatible with v, the image of $P \cap A$ under the projection $A \to \bar{F}$ gives a well-defined ordering $\bar{P}$ on $\bar{F}$. We shall denote the orderings compatible with v by X_F^v.

THEOREM 2. *Let (K, P) be an ordered field. For any subfield $F \subseteq K$. let $A(F, P)$ be the convex hull of F with respect to P. Then we have $A(F, P) = \{a \in K \mid \exists b \in F$ such that $-b \underset{P}{\leqslant} a \underset{P}{\leqslant} b\}$ and $A(F, P)$ is a valuation ring of K. The unique maximal ideal $I(F, P)$ of $A(F, P)$ consists of infinitely small elements of K* [4].

A valuation is called real if its residue class field is a formally real field. The following theorem by Baer and Krull [1], [3] is as crucial as Theorem 2 in our proof of main theorem.

THEOREM 3. *Let v be a real valuation of F. Then any ordering Q on $\bar{F}$ can be lifted to an ordering on F. That is, there exists $P \in X_F^v$ such that $\bar{P} = Q$. See* (4).

We can now prove our theorem. We begin with an easy lemma.

LEMMA. *Let K, K' be extension fields of a field F. If there exists an F-place $\varphi: K \to K' \cup \{\infty\}$, then the residue class field $\bar{K}$ of the associated valuation of φ satisfies the relation $F \subseteq \bar{K} \subseteq K'$, where the inclusions are obtained by identifications.*

Proof. This is obtained by a simple consideration of valuation theory.

MAIN THEOREM. *Let (F, P_0) be an ordered field, and K an extension field of F. Then $P_0 \in X_F$ can be extended to an ordering P on K iff there exists an F-place φ of K into some real closed field R with $R \subseteq F$ and $R^2 \cap F = P_0$.*

Proof. (The only if part) Since $P_0 \in X_F$ is extended to $P \in X_K$, (K, P) is an ordered field extension of (F, P_0). By Theorem 2 we have a natural valuation ring $A(F, P) \supseteq F$ and the associated place $\pi: K \to \bar{K}$ $(=A(F, P)/I(F, P)) \cup \{\infty\}$, So we can identify F as a subfield of $\bar{K}$ by definition of $A(F, P)$.

Then π becomes an F-place, and $(\bar{K}, \bar{P})$ is an ordered field extension of (F, P_0). Denoting a real closure of $(\bar{K}, \bar{P})$ by R, we get the desired F-place φ by compositing the associated place π with the inclusion of $(\bar{K}, \bar{P})$ into R.

(The if part) Assume that there exists an F-place φ of K into some real closed field R with $R \subseteq F$ and $R^2 \cap F = P_0$. Then we have $F \subseteq \bar{K} \subseteq R$ by the lemma, where $\bar{K}$ is the residue class field of the associated valuation v of φ. If $R^2 \cap \bar{K} = Q \in X_{\bar{K}}$, then the tower of fields $(F, P_0) \subseteq (\bar{K}, Q) \subseteq (R, R^2)$ becomes a tower of ordered fields. The ordering Q on $\bar{K}$ can be lifted to $P \in X_K^v$ by Theorem 3. Then we have $P \cap F = \overline{P \cap F} = \bar{P} \cap F = Q \cap F = P_0$, i.e, $P \in X_K$ extends the given ordering P_0 on F.

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