SOME GENERALIZATION OF THE LANG’S EXISTENCE OF RATIONAL PLACE THEOREM

IN-HO CHO AND JONG IN LIM.

1. Introduction

Let $K$ be a real function field over a real closed field $F$. Then there exists an $F$-place $\varphi : K \to F \cup \{\infty\}$. This is Lang’s Existence of Rational Place Theorem (6). There is an equivalent version of Lang’s Theorem in (4). That is, if $K$ is a function field over a field $F$, then, for any ordering $P_0$ on $F$ which extends to $K$, there exists an $F$-place $\varphi : K \to F' \cup \{\infty\}$ where $F'$ is a real closure of $(F, P_0)$.

In [2], Knebusch pointed out the converse of the version of Lang’s Theorem is also true.

By a valuation theoretic approach to Lang’s Theorem, we have found out the following generalization of Lang and Knebusch’s Theorem. Let $K$ be an arbitrary extension field of a field $F$. Then an ordering $P_0$ on $F$ can be extended to an ordering $P$ on $K$ iff there exists an $F$-place of $K$ into some real closed field $R$ containing $F$. Of course $R_0 \cap F = P_0$. The restriction $K$ being a function field of $F$ is vanished, though the codomain of the $F$-place is slightly varied. Therefore our theorem is a generalization of Lang and Knebusch’s theorem.

2. Preliminaries and Main Theorem

By an ordering on a field $F$, we mean a subset $P \subseteq F$ such that $P + P \subseteq P$, $P \cdot P \subseteq P$ and $P \cup (-P) = F$. From these axioms, it is easy to see that $P \cap (-P) = \{0\}$. [4].

The set of all orderings on $F$ will be denoted by $X_F$. If $P \in X_F$, then the pair $(F, P)$ is called an ordered field. For an extension field $K$ of $F$, an ordering $Q \in X_K$ is said to extend $P \in X_F$, if $Q \cap F = P$. By a valuation on a field $F$, we shall always mean Krull valuation $v : F \to \Gamma$ onto an ordered group $\Gamma$, satisfying the two axioms

\begin{enumerate}
  \item $v(xy) = v(x) + v(y)$ for any $x, y \in \hat{F} = F \setminus \{0\}$,
  \item $v(x + y) \geq \min\{v(x), v(y)\}$ for $x, y, x + y \in \hat{F}$.
\end{enumerate}

For a given valuation $v$ as above, we can define the following collection of associated objects.

$A : = \{x \in F \mid x = 0 \text{ or } v(x) \geq 0\}$ (the valuation ring of $v$),
$\mathfrak{m} : = \{x \in F \mid x = 0 \text{ or } v(x) > 0\}$ (the maximal ideal of $v$),
$U : = A \setminus \mathfrak{m}$ (the group of valuation units),

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\[ \mathcal{P} := \mathbb{A}/\mathfrak{m} \text{ (the residue class field of } v), \]
\[ \pi : \mathbb{F} \rightarrow \mathcal{P} \cup \{\infty\} \text{ denote the place associated with } v, \]
\[ \text{defined by } \pi(x) = \begin{cases} x + \mathfrak{m} & \text{if } x \in \mathbb{A} \\ \infty & \text{if } x \in \mathbb{A}. \end{cases} \]

We usually write \( \bar{x} \) for \( x + \mathfrak{m} \), and say \( (v, \mathbb{A}, \mathfrak{m}, \mathbb{P}, \cdots) \) is a valuation instead of saying \( v \) is a valuation. We shall write \( a \succcurlyeq_P b \) if \( a - b \in \mathbb{P} \) and \( a \succcurlyeq b \) if \( a - b \in \mathbb{P} = \mathbb{P} \setminus \{0\} \).

**Theorem 1.** Let \( P \subseteq X_F \), and \( (v, \mathbb{A}, \mathfrak{m}, \mathbb{P}, \cdots) \) be a valuation on \( F \). Then the following statements are equivalent (5).

1. \( 0 \preccurlyeq_P a \leq_P b \Rightarrow v(a) \geq v(b) \) in \( \mathbb{P} \).
2. \( \mathbb{A} \) is convex with respect to \( P \).
3. \( \mathfrak{m} \) is convex with respect to \( P \).
4. \( 1 + \mathfrak{m} \subseteq P \).

**Definition 1.** If any (and hence all) of the conditions in Th. 1 holds for \( v \) and \( P \), we shall say \( v \) is compatible with \( P \) (or that \( P \) is compatible with \( v \)).

In case \( P \) is compatible with \( v \), the image of \( P \cap \mathbb{A} \) under the projection \( \mathbb{A} \rightarrow \bar{\mathbb{F}} \) gives a well-defined ordering \( \bar{P} \) on \( \bar{\mathbb{F}} \). We shall denote the orderings compatible with \( v \) by \( X_F^\nu \).

**Theorem 2.** Let \( (K, P) \) be an ordered field. For any subfield \( F \subseteq K \), let \( A(F, P) \) be the convex hull of \( F \) with respect to \( P \). Then we have \( A(F, P) = \{a \in K \mid \exists b \in F \text{ such that } -b \leq_P a \leq_P b \} \) and \( A(F, P) \) is a valuation ring of \( K \). The unique maximal ideal \( I(F, P) \) of \( A(F, P) \) consists of infinitely small elements of \( K \) [4].

A valuation is called real if its residue class field is a formally real field. The following theorem by Baer and Krull [1], [3] is as crucial as Theorem 2 in our proof of main theorem.

**Theorem 3.** Let \( v \) be a real valuation of \( F \). Then any ordering \( Q \) on \( \bar{\mathbb{F}} \) can be lifted to an ordering on \( F \). That is, there exists \( P \subseteq X_F^\nu \) such that \( \bar{P} = Q \).
See (4).

We can now prove our theorem. We begin with an easy lemma.

**Lemma.** Let \( K, K' \) be extension fields of a field \( F \). If there exists an \( F \)-place \( \varphi : K \rightarrow K' \cup \{\infty\} \), then the residue class field \( \bar{K} \) of the associated valuation of \( \varphi \) satisfies the relation \( F \subseteq \bar{K} \subseteq K' \), where the inclusions are obtained by identifications.

**Proof.** This is obtained by a simple consideration of valuation theory.

**Main Theorem.** Let \( (F, P_0) \) be an ordered field, and \( K \) an extension field of \( F \). Then \( P_0 \subseteq X_F \) can be extended to an ordering \( P \) on \( K \) iff there exists an \( F \)-place \( \varphi \) of \( K \) into some real closed field \( R \) with \( R \subseteq F \) and \( R^2 \cap F = P_0 \).
Some generalization of the Lang's existence of rational place Theorem

Proof. (The only if part) Since $P_0 \subseteq X_F$ is extended to $P \subseteq X_K$, $(K, P)$ is an ordered field extension of $(F, P_0)$. By Theorem 2 we have a natural valuation ring $A(F, P) \supseteq F$ and the associated place $\pi: K \rightarrow \bar{K} (= A(F, P)/I(F, P)) \cup \{\infty\}$. So we can identify $F$ as a subfield of $\bar{K}$ by definition of $A(F, P)$.

Then $\pi$ becomes an $F$-place, and $(\bar{K}, \bar{P})$ is an ordered field extension of $(F, P_0)$. Denoting a real closure of $(\bar{K}, \bar{P})$ by $R$, we get the desired $F$-place $\varphi$ by composing the associated place $\pi$ with the inclusion of $(\bar{K}, \bar{P})$ into $R$.

(The if part) Assume that there exists an $F$-place $\varphi$ of $K$ into some real closed field $R$ with $R \subseteq F$ and $R^2 \cap F = P_0$. Then we have $F \subseteq \bar{K} \subseteq R$ by the lemma, where $\bar{K}$ is the residue class field of the associated valuation $v$ of $\varphi$. If $R^2 \cap \bar{K} = Q \subseteq X_{\bar{K}}$, then the tower of fields $(F, P_0) \subseteq (\bar{K}, Q) \subseteq (R, R^2)$ becomes a tower of ordered fields. The ordering $Q$ on $\bar{K}$ can be lifted to $P \subseteq X_{\bar{K}}$ by Theorem 3. Then we have $P \cap F = P \cap \bar{F} = P \cap F = Q \cap F = P_0$, i.e., $P \subseteq X_K$ extends the given ordering $P_0$ on $F$.

References


Korea University
Seoul 132, Korea