Bull. Korean Math. Soc. 22 (1985), No. 2, pp. 89-93

ON HOMOMORPHISMS ON C^*-ALGEBRAS

TAE-GEUN CHO

1. Introduction

One of the most important problems in automatic continuity theory is to solve the question of continuity of an algebra homomorphism from a Banach algebra into a semisimple Banach algebra with dense range. Many results on this subject are obtained imposing some conditions on the domains or the ranges of homomorphisms. For most recent results and references in automatic continuity theory one may refer to [1], [4] and [5].

In this note we study some properties of homomorphisms from C^*-algebras into Banach algebras. It is shown that the range of an isomorphism from a C^*-algebra into a Banach algebra contains no non zero element of the radical of B. Using this result we show that the same holds for a continuous homomorphism, hence a Banach algebra which is the image of a C^*-algebra under a continuous homomorphism is necessarily semisimple. Thus if there is a homomorphism from a C^*-algebra onto a non-semisimple Banach algebra it must be discontinuous . Also it follows that every non zero homomorphism from a C^*-algebra into a radical algebra is discontinuous. Then we make a brief observation on the behavior of quasi-nilpotent elements of noncommutative C^*-algebras in relation with continuous homomorphisms.

2. Definitions and Basic Facts

In this section we present some definitions and basic facts which we need in proving the theorems in the next section.

1. DEFINITION. The (Jacobson) *radical* of a Banach algebra A is the intersection of all maximal modular left ideals in A if such ideals exist, is the algebra A itself if there is no maximal modular left ideal in A. The radical of A is denoted by rad(A). The Banach algebra A is called *semisimple* if rad(A) contains only the zero element of A, on the other hand A is called a *radical algebra* if rad(A) is A itself.

By the definition of the radical of a Banach algebra, the radical is a closed left ideal, but it is in fact a closed bi-ideal of the algebra ([2], p. 125).

2. DEFINITION. An element a in a Banach algebra is said to be *quasi-nilpotent*

This research is supported by MOE grant, 1984.

if $\lim\|a^n\|^{\frac{1}{n}}=0$.

Since $\lim\|a^n\|^{\frac{1}{n}}$ is the spectral radius of a, an element a is quasi-nilpotent if and only if the spectral radius of a is equal to zero, that is the spectrum of a contains a single element 0.

In a Banach algebra every element of the radical is quasi-nilpotent([2], p. 126). But in a noncommutative Banach algebra a quasi-nilpotent element need not be in the radical. It is known that every noncommutative C^*-algebra contains a non zero nilpotent element and this element does not belong to the radical of the algebra since every C^*-algebra is semisimple.

In a commutative Banach algebra the radical coincides with the set of quasi-nilpotent elements. In fact, if a is a quasi-nilpotent element in a commutative Banach algebra A then we have

$$\hat{a}(\Phi_A)=\sigma(a)=\{0\},$$

here, $a\rightarrow\hat{a}$ is the Gelfand representation of A, Φ_A the carrier space of A and $\sigma(a)$ denotes the spectrum of a. Thus, for each character f in Φ_A $f(a)=0$, that is, a belongs to the kernel of each f. But the kernel of a character is a maximal modular ideal of A, consequently a belongs to the radical of A.

A monomorphism θ from a commutative C^*-algebra A into a Banach algebra B is norm increasing [3]. Hence, if a is a non zero element of A, then for each positive integer n

$$\|\theta(a)^n\|\geq\|a^n\|=\|\hat{a}\|_\infty^n>0.$$

Therefore $\theta(a)$ is not quasi-nilpotent and $\theta(a)$ does not belong to the radical of B. Thus we have

$$\mathrm{rad}(B)\cap\theta(A)=\{0\}.$$

We will show that this equality holds for a monomorphism and a continuous homomorphism on a noncommutative C^*-algebra.

3. Homomorphism on C^*-algebras

1. THEOREM. *Let A be a C^*-algebra and B be a Banach algebra. Then for each monomorphism θ from A into B we have*

$$rad(B)\cap\theta(A)=\{0\}.$$

Proof. Let $\theta:A\longrightarrow B$ be a monomorphism from A into B and let b be an arbitrary element of $\mathrm{rad}(B)\cap\theta(A)$, then there exists an element a in A with $\theta(a)=b$. Let A' be the C^*-algera generated by the self adjoint element $a'=aa^*$ and let B' be the closure of $\theta(A')$ in B, then θ restricted to the commutative C^*-algebra A' is a monomorphism which maps A' into B'. Therefore we have

$$\mathrm{rad}(B')\cap\theta(A')=\{0\}.$$

On the other hand, $\mathrm{rad}(B)$ is a bi-ideal of B and $b\theta(a^*)$ belongs to $\mathrm{rad}(B)$. Hence $b\theta(a^*)$ is a quasi-nilpotent element of B. But we have

$$b\theta(a^*)=\theta(a)\theta(a^*)=\theta(a')\in\theta(A'),$$

and since B' is a commutative Banach algebra, $b\theta(a^*)$ belongs to $\mathrm{rad}(B')$. Therefore we have

$$b\theta(a^*)\in\mathrm{rad}(B')\cap\theta(A')=\{0\},$$

whence $\theta(aa^*)=b\theta(a^*)=0$. Since θ is a monomorphism we have $aa^*=0$ and $a=0$. Hence $b=\theta(a)=0$, and we have proved

$$\mathrm{rad}(B)\cap\theta(A)=\{0\}.$$

The following well-known result is a simple consequence of Theorem 1.

2. COROLLARY. *Every C^*-algebra is semisimple.*

Proof. Let θ be the identity map on a C^*-algebra A. Then

$$\mathrm{rad}(A)\ =\mathrm{rad}(A)\cap\theta(A)=\{0\}.$$

3. COROLLARY. *Every monomorphism from a (noncommutative) C^*-algebra onto a Banach algebra is continuous.*

Proof. The range of the monomorphism is a semisimple Banach algebra by Theorem 1. But every epimorphism from a Banach algebra onto a semisimple Banach algebra is continuous ([5], Theorem 6.12), hence the monomorphism is continuous.

REMARK. This corollary can be also proved by considering the inverse map of θ applying Theorem 6.12 of [5].

4. THEOREM. *Let A be a C^*-algebra and B be a Banach algebra. Then, for each continuous homomorphism θ from A into B we have*

$$\mathrm{rad}(B)\cap\theta(A)=\{0\}.$$

Proof. If $\theta:A\longrightarrow B$ is a continuous homomorphism from A itno B, then $\ker(\theta)$, the kernel of θ, is a closed bi-ideal of the C^*-algebra A and it is self adjoint ([6], Theorem I. 8.1). Thus θ induces a continuous monomorphism

$$\bar{\theta}:A/\ker(\theta)\longrightarrow B$$

such that $\theta=\bar{\theta}\circ\pi$ where $\pi:A\longrightarrow A/\ker(\theta)$ is the canonical quotient map. Since $A/\ker(\theta)$ is a C^*-algebra, applying Theorem 1 to the monomorphism $\bar{\theta}$, we have

$$\mathrm{rad}(B)\cap\theta(A)=\mathrm{rad}(B)\cap\bar{\theta}(A/\ker(\theta))=\{0\}.$$

Since every closed subalgebra of a radical algebra is radical we have the following corollary.

5. COROLLARY. *Every non zero homomorphism from a C^*-algebra into a radical Banach algebra is discontinuous.*

6. COROLLARY. *Let A be a C^*-algebra and B be a non-semisimple Banach algebra. If there is an epimorphism $\theta:A\longrightarrow B$ from A onto B then it must be*

discontinuous.

Proof. If an epimorphism $\theta: A \longrightarrow B$ is continuous, then we have

$$\operatorname{rad}(B) = \operatorname{rad}(B) \cap \theta(A) = \{0\}.$$

Hence B must be semisimple.

By Theorem 1 a Banach algebra which is the image of a C^*-algebra under a monomorphism must be semisimple. At present we do not know whether the same holds for an epimorphism.

Since the radical of a commutative Banach algebra coincides with the set of quasi-nilpotent elements, a commutative semisimple Banach algebra has no non zero quasi-nilpotent element. There are noncommutative Banach algebras which contain no non zero qusai-nilpotent element. An example of such an algebra constructed by Duncan and Tullo is presented in [2, p. 254]

7. THEOREM. *A homomorphism θ from a C^*-algebra into a Banach algebra which contains no non zero quasi-nilpotent element is continuous, and the kernel of such a homomorphism contains all the quasi-nilpotent elements of A.*

Proof. Let a be a self adjoint element of a and let A' be the C^*-algebra generated by the element a. Then the closure of $\theta(A')$ in B is a commutative semisimple Banach algebra since zero is the only quasi-nilpotent element. Thus θ is continuous on A' and hence θ is continuous on A by Theorem 12.7 of [5]. The second assertion is clear since the image of a quasi-nilpotont element of A is also quasi-nilpotent.

If B is a Banach algebra in which the radical coincides with the set of quasi-nilpotent elements, then the kernel of a continuous homomorphism θ from a C^*-algebra into B contains all the quasi-nilpotent elements of A. For, if a is a quasi-nilpotent element of A, then $\theta(a)$ is also a quasi-nilpotent element of B and it belongs to $\operatorname{rad}(B)$, but by Theorem 4 we have

$$\operatorname{rad}(B) \cap \theta(A) = \{0\}.$$

References

1. J.M. Bachar, W.G. Badé, P.C. Curtis, Jr., H.G. Dales and M.P. Thomas, *Radical Banach Algebras and Automatic Continuity*, Lecture Note in Mathematic **975**, Springer-Verlag, New York (1983).
2. F.F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York (1973).
3. T.G. Cho, *Continuity of Homomorphisms between Banach Algebras*, Bull. Korean Math. Soc. **20**, No. 2 (1983), 71-74.
4. H.G. Dales, *Automatic Continuity*; a survey, Bull. London Math. Soc. **10** (1978), 129-183.

5. A.M. Sinclair, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note series Vol. **21**, Cambridge University Press, Cambridge (1979).

6. M. Takesaki, *Theory of Operator Algebras* I, Springer-Verlag, New York (1979).

Sogang University
Seoul 121, Korea