ON THE DOMAIN OF NULL-CONTROLLABILITY OF A LINEAR PERIODIC SYSTEM

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0. Introduction

In [1], E.B. Lee and L. Markus described a sufficient condition for which the domain of null-controllability of a linear autonomous system is all of $\mathbb{R}^n$. The purpose of this note is to extend the result to a certain linear nonautonomous system. Thus we consider a linear control system

$$\frac{dx}{dt} = A(t)x + B(t)u$$

in the Euclidean $n$-space $\mathbb{R}^n$ where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, respectively, which are continuous on $0 \leq t < \infty$ and $A(t)$ is a periodic matrix of period $\omega$. Admissible controls are bounded measurable functions defined on some finite subintervals of $[0, \infty)$ having values in a certain convex set $\Omega$ in $\mathbb{R}^m$ with the origin in its interior. And we present a sufficient condition for which the domain of null-controllability is all of $\mathbb{R}^n$.

1. Preliminaries

Consider a linear control system in $\mathbb{R}^n$

$$(1.1) \quad \frac{dx}{dt} = A(t)x + B(t)u$$

Where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, respectively, which are continuous on $[0, \infty)$ but not necessarily periodic.

For any bounded measurable control $u(t)$ on $[t_0, t_1]$ ($t_0 \geq 0$) and for any initial state $x_0$ at $t_0$, (1.1) has a unique solution $x(t)$ with $x(t_0) = x_0$ existing on $[t_0, t_1]$ and this solution is given by

$$(1.2) \quad x(t) = F(t_0; t)x_0 + \int_{t_0}^{t} F^{-1}(t_0; s)B(s)u(s)ds$$

where $F(t_0; t)$ is a fundamental matrix of the homogeneous system

$$\frac{dx}{dt} = A(t)x$$

with $F(t_0; t_0) = I$ (the identity matrix).

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**Definition 1.** We say that the system (1.1) is completely controllable at \( t_0 \geq 0 \) if, for any \( x_0, x_1 \) in \( R^n \) there exists a bounded measurable control \( u(t) \) on some finite interval \([t_0, t_1]\) with values in \( R^n \) such that

\[
x_1 = F(t_0; t_1)x_0 + F(t_0; t_1) \int_{t_0}^{t_1} F^{-1}(t_0; s) B(s) u(s) \, ds,
\]

that is, there exists a bounded measurable control on some finite interval \([t_0, t_1]_c which steers \( x_0 \) to \( x_1 \).

It is well known that the system (1.1) is completely controllable at \( t_0 \) iff, for any \( t_1 > t_0 \), the matrix

\[
M(t_0, t_1) = \int_{t_0}^{t_1} F^{-1}(t_0; t) B(t) B'(t) F^{-1'}(t_0; t) \, dt
\]

is nonsingular where the prime denotes the transposed matrix.

**2. Domain of Null-Controllability**

Let \( \Omega \) be a given convex set in \( R^n \) containing the origin in its interior and consider a linear control system in \( R^n \)

\[
(2.1) \quad \frac{dx}{dt} = A(t)x + B(t)u, \quad u \in \Omega
\]

where \( A(t) \) and \( B(t) \) are same as in (1.1) and admissible controls are bounded measurable functions on some finite subinterval of \([0, \infty) \) having values in \( \Omega \).

**Definition 2.** The domain of null-controllability of (2.1) at \( t_0 \geq 0 \) is the set \( C(t_0) \) of those points \( x_0 \) in \( R^n \) which can be steered to the origin by some admissible control; that is, there exists a bounded measurable control \( u(t) \) on some interval \([t_0, t_1]\) having values in \( \Omega \) such that

\[
F(t_0; t_1)x_0 + F(t_0; t_1) \int_{t_0}^{t_1} F^{-1}(t_0; s) B(s) u(s) ds = 0
\]
equivalently,

\[
x_0 = -\int_{t_0}^{t_1} F^{-1}(t_0; s) B(s) u(s) \, ds
\]

It is clear that \( C(t_0) \) is convex and \( 0 \in C(t_0) \).

**Theorem 1.** The domain \( C(t_0) \) of null-controllability at \( t_0 \geq 0 \) contains a neighborhood of the origin iff the system (2.1) is completely controllable at \( t_0 \).

**Proof.** For \( t_1 > t_0 \), let

\[
M(t_0, t_1) = \int_{t_0}^{t_1} F^{-1}(t_0; t) B(t) B'(t) F^{-1'}(t_0; t) \, dt
\]

Then \( M(t_0, t_1) \) is a symmetric matrix which is positive semidefinite.

Suppose that the system (2.1) is completely controllable at \( t_0 \). Then \( M(t_0, t_1) \) is nonsingular for any \( t_1 > t_0 \). Choose any \( t_1 > t_0 \). Since \( F^{-1}(t_0; t) \) and \( B(t) \) are
continuous on \([t_0, t_1]\), there exists a constant \(K_1 > 0\) such that \(|F^{-1}(t_0; t)| \leq K_1\) and \(|B(t)| \leq K_1\) for all \(t_0 \leq t \leq t_1\). Let

\[|M(t_0, t_1)| = K_2 > 0.\]

Choose \(r > 0\) so that \(|u| < r\) implies \(u \in \Omega\). Let \(x\) be any point in \(\mathbb{R}^n\) such that \(|x| < \frac{r}{K_1^2 K_2}\). If we let, for \(t_0 \leq t \leq t_1\),

\[\xi = -M(t_0, t_1)^{-1} x, \quad u(t) = B'(t) F^{-1'}(t_0; t) \xi\]

then

\[|u(t)| = |B'(t) F^{-1'}(t_0; t) \xi| \leq K_1^2 K_2 |x| < r\]

so that \(u(t) \in \Omega\) for all \(t_0 \leq t \leq t_1\). Moreover,

\[- \int_{t_0}^{t_1} F^{-1}(t_0; t) B(t) u(t) dt = - \int_{t_0}^{t_1} F^{-1}(t_0; t) B(t) B'(t) F^{-1'}(t_0; t) \xi dt\]

\[= -M(t_0, t_1) \xi = x.\]

Thus \(x \in C(t_0)\); that is \(C(t_0)\) contains the set

\[\left\{x \in \mathbb{R}^n : |x| < \frac{r}{K_1^2 K_2}\right\}\]

Conversely, suppose \(C(t_0)\) does not contain a neighborhood of the origin. Then \(C(t_0)\) lies on a hyperplane passing through the origin. Thus there exists a nonzero vector \(\xi \in \mathbb{R}^n\) such that, for any \(t_1 > t_0\) and for any admissible control \(u(t)\) on \([t_0, t_1]\),

\[\int_{t_0}^{t_1} \xi F^{-1}(t_0; t) B(t) u(t) dt = 0.\]

For any \(t_1 > t_0\), consider the control \(u(t) = B'(t) F^{-1'}(t_0; t) \xi'\) on \([t_0, t_1]\). For \(|\xi|\) sufficiently small \(u(t) \in \Omega\) for all \(t_0 \leq t \leq t_1\), and we must have

\[\int_{t_0}^{t_1} \xi F^{-1}(t_0; t) B(t) B'(t) F^{-1'}(t_0; t) \xi' dt = -\xi M(t_0, t_1) \xi' = 0.\]

Thus \(M(t_0, t_1)\) is singular so that system (2.1) is not completely controllable at \(t_0\).

3. Linear Periodic System

Now consider a linear periodic system in \(\mathbb{R}^n\)

\[
\frac{dx}{dt} = A(t) x - B(t) u, \quad u \in \Omega
\]

where \(A(t), B(t)\) and \(\Omega\) are same as in (2.1) and, in addition, we assume that \(A(t)\) is a periodic matrix of period \(\omega\) on \([0, \infty)\).

If \(F(t)\) is a fundamental matrix of the corresponding homogeneous system

\[
\frac{dx}{dt} = A(t) x
\]
then, by Floquet's theorem, there exists a periodic nonsingular matrix $P(t)$ of period $\omega$ and a constant matrix $R$ such that $F(t) = P(t) \exp(tR)$. We call the eigenvalues of the nonsingular matrix $\exp(\omega R)$ the multipliers of the system (3.2) and the eigenvalues of the matrix $R$ are called the characteristic exponents of the system (3.2).

Following lemma is well known.

**Lemma.** If all the characteristic exponents of the system (3.2) have negative real parts, then the zero solution of (3.2) is asymptotically stable.

Combining Theorem 1 and the above lemma, we obtain the following theorem which is the main result.

**Theorem 2.** For the linear periodic system (3.1), suppose
(1) $0$ is in the interior of $\Omega$
(2) system (3.1) is completely controllable at every $t' \geq t_0$
(3) characteristic exponents of (3.2) have negative real parts.

Then the domain $C(t_0)$ of null-controllability of (3.1) at $t_0$ is all of $\mathbb{R}^n$.

**Proof.** Let $x_0$ be any point in $\mathbb{R}^n$ and choose the control $u(t)$ which identically zero for all $t \geq 0$. If $x(t)$ is the solution of (3.1) corresponding to $u(t)$ with $x(t_0) = x_0$, then, by condition (3), $x(t')$ belongs to $C(t')$ for sufficiently large $t'$. But then there exists an admissible control $v(t)$ on some interval $[t', t_1]$ which steers $x(t')$ to the origin.

**References**


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