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CERTAIN CLASSES OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS+

SHIGEYOSHI OWA AND H. M. SRIVASTAVA

1. Introduction

Let $\mathcal{T}_p$ denote the class of functions defined by

$$(1.1)\qquad f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad (a_{p+n}\geqq 0;\ p\in\mathcal{N}=\{1,2,3,\cdots\}),$$

which are analytic and p-valent in the unit disk $\mathcal{U}=\{z:|z|<1\}$. For α and β fixed, $-1\leqq\alpha<\beta\leqq 1$, $0<\beta\leqq 1$, we say that $f(z)$ is in the subclass $\mathcal{T}^*(p,\alpha,\beta)$ of $\mathcal{T}_p$ if and only if

$$(1.2)\qquad \left|\left[\frac{f'(z)}{z^{p-1}}-p\right]\Big/\left[\frac{\beta f'(z)}{z^{p-1}}-\alpha p\right]\right|<1\qquad (z\in\mathcal{U}).$$

Further, $f(z)$ is said to be in the subclass $\mathcal{C}(p,\alpha,\beta)$ of $\mathcal{T}_p$ if and only if

$$(1.3)\qquad zf'(z)/p\in\mathcal{T}^*(p,\alpha,\beta).$$

The class $\mathcal{T}^*(p,\alpha,\beta)$ was studied recently by Shukla and Dashrath [9]. The subclasses $\mathcal{T}_p^*(\alpha,\beta)$ and $\mathcal{C}_p(\alpha,\beta)$ of $\mathcal{T}_p$ obtained by replacing $f'(z)/z^{p-1}$ in (1.2) by $zf'(z)/f(z)$ and $1+zf''(z)/f'(z)$, respectively, have been studied by Goel and Sohi [2], and by Srivastava and Owa [11].

In the present paper, we give (i) sharp results involving the coefficient estimates of functions $f(z)$ in the class $\mathcal{C}(p,\alpha,\beta)$, (ii) the extremal properties of the class $\mathcal{C}(p,\alpha,\beta)$, (iii) several useful properties of $\mathcal{T}^*(p,\alpha,\beta)$ and $\mathcal{C}(p,\alpha,\beta)$, and (iv) closure theorems and other interesting results involving the modified Hadamard product of two functions belonging to the classes $\mathcal{T}^*(p,\alpha,\beta)$ and $\mathcal{C}(p,\alpha,\beta)$. As applications of the aforementioned distortion theorems, we also prove general theorems on the *fractional* calculus of functions $f(z)$ belonging to the classes $\mathcal{T}(p,\alpha,\beta)$ and $\mathcal{C}(p,\alpha,\beta)$.

2. Coefficient Estimates

For the class $\mathcal{T}^*(p,\alpha,\beta)$, Shukla and Dashrath[9] gave the following lemma

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which will be required in our derivation of the coefficient estimates for functions belonging to the class $\mathscr{C}(p, \alpha, \beta)$.

LEMMA 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z)$ is in the class $\mathscr{T}^*(p, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n)(1+\beta)a_{p+n} \leqq (\beta-\alpha)p. \tag{2.1}$$

The result (2.1) *is sharp.*

By using Lemma 1, we shall now prove

THEOREM 1. *Let the function $f(z)$ be defined by* (1.1). *Then $f(z)$ is in the class $\mathscr{C}(p, \alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n)^2(1+\beta)a_{p+n} \leqq (\beta-\alpha)p^2. \tag{2.2}$$

The result (2.2) *is sharp*

Proof. The function $f(z)$ is in the class $\mathscr{C}(p, \alpha, \beta)$ if and only if (1.3) holds true. Since

$$\frac{zf'(z)}{p} = z^p - \sum_{n=1}^{\infty} \left[\frac{p+n}{p}\right] a_{p+n} z^{p+n}, \tag{2.3}$$

by replacing a_{p+n} by $\left(\frac{p+n}{p}\right) a_{p+n}$ in Lemma 1, we have the theorem. Moreover, for the function given by

$$f(z) = z^p - \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} z^{p+n} \qquad (n \in \mathscr{N}), \tag{2.4}$$

we can see that the assertion (2.2) is sharp.

COROLLARY 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathscr{C}(p, \alpha, \beta)$. Then*

$$a_{p+n} \leqq \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} \qquad (n \geqq 1). \tag{2.5}$$

The equality in (2.5) *holds true for the function $f(z)$ given by* (2.4).

We next prove the following distortion theorem with the aid of Theorem 1.

THEOREM 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathscr{C}(p, \alpha, \beta)$. Then*

$$|f(z)| \geqq |z|^p - \frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|^{p+1}, \tag{2.6}$$

$$|f(z)| \leqq |z|^p + \frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|^{p+1}, \tag{2.7}$$

$$|f'(z)| \geqq p|z|^{p-1} - \frac{(\beta-\alpha)p^2}{(p+1)(1+\beta)}|z|^p, \tag{2.8}$$

and

(2.9) $$|f'(z)|\leqq p|z|^{p-1}+\frac{(\beta-\alpha)p^2}{(p+1)(1+\beta)}|z|^p$$

for $z\in\mathscr{U}$. *Furthermore, if* $p\in\mathscr{N}-\{1\}$, *then*

(2.10) $$|f''(z)|\geqq p(p-1)|z|^{p-2}-\frac{(\beta-\alpha)p^2}{1+\beta}|z|^{p-1}$$

and

(2.11) $$|f''(z)|\leqq p(p-1)|z|^{p-2}+\frac{(\beta-\alpha)p^2}{1+\beta}|z|^{p-1}$$

for $z\in\mathscr{U}$

The estimates for $|f(z)|$ *and* $|f'(z)|$ *are sharp, and are attained for the function*

(2.12) $$f(z)=z^p-\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}z^{p+1}.$$

Proof. In view of Theorem 1, we have

(2.13) $$(p+1)^2(1+\beta)\sum_{n=1}^{\infty}a_{p+n}\leqq\sum_{n=1}^{\infty}(p+n)^2(1+\beta)a_{p+n}\leqq(\beta-\alpha)p^2,$$

which implies that

(2.14) $$\sum_{n=1}^{\infty}a_{p+n}\leqq\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}$$

Hence

(2.15) $$\begin{aligned}|f(z)|&\geqq|z|^p-\sum_{n=1}^{\infty}a_{p+n}|z|^{p+n}\\&\geqq|z|^p-|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}\\&\geqq|z|^p-\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|^{p+1}\end{aligned}$$

and

(2.16) $$\begin{aligned}|f(z)|&\leqq|z|^p+\sum_{n=1}^{\infty}a_{p+n}|z|^{p+n}\\&\leqq|z|^p+|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}\\&\leqq|z|^p+\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|^{p+1}\end{aligned}$$

for $z\in\mathscr{U}$.

Next we observe that

(2.17) $$\sum_{n=1}^{\infty}(p+n)a_{p+n}\leqq\frac{(\beta-\alpha)p^2}{(p+1)(1+\beta)}$$

Consequently, we can prove (2.8) and (2.9) by applying (2.17) in the same manner as in the proof of (2.6) and (2.7).

Finally, for $p\in\mathscr{N}-\{1\}$ and $z\in\mathscr{U}$,

$$(2.18)\quad |f''(z)| \geqq p(p-1)|z|^{p-2} - \sum_{n=1}^{\infty}(p+n)(p+n-1)a_{p+n}|z|^{p+n-2}$$
$$\geqq p(p-1)|z|^{p-2} - |z|^{p-1}\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}$$
$$\geqq p(p-1)|z|^{p-2} - \frac{(\beta-\alpha)p^2}{1+\beta}|z|^{p-1}$$

and

$$(2.19)\quad |f''(z)| \leqq p(p-1)|z|^{p-2} + \sum_{n=1}^{\infty}(p+n)(p+n-1)a_{p+n}|z|^{p+n-2}$$
$$\leqq p(p-1)|z|^{p-2} + |z|^{p-1}\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}$$
$$\leqq p(p-1)|z|^{p-2} + \frac{(\beta-\alpha)p^2}{1+\beta}|z|^{p-1},$$

in view of Theorem 1.

COROLLARY 2. *Under the hypotheses of Theorem 2, $f(z)$ is included in the disk with center at the origin and radius r_1 given by*

$$(2.20)\quad r_1 = 1 + \frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)},$$

$f'(z)$ is included in the disk with center at the origin and radius r_2 given by

$$(2.21)\quad r_2 = p + \frac{(\beta-\alpha)p^2}{(p+1)(1+\beta)},$$

and $f''(z)$ is included in the disk with center at the origin and radius r_3 given by

$$(2.22)\quad r_3 = p(p-1) + \frac{(\beta-\alpha)p^2}{1+\beta}$$

3. Properties of $\mathscr{T}^*(p, \alpha, \beta)$ and $\mathscr{C}(p, \alpha, \beta)$

In this section we derive some useful properties of the classes $\mathscr{T}^*(p, \alpha, \beta)$ and $\mathscr{C}(p, \alpha, \beta)$ by employing Lemma 1 and Theorem 1.

THEOREM 3. *Let $-1 \leqq \alpha_1 \leqq \alpha_2 \leqq 1$, $0 < \beta_1 \leqq \beta_2 \leqq 1$, and $\alpha_2 < \beta_1$. Then*

$$(3.1)\quad \mathscr{T}^*(p, \alpha_1, \beta_2) \supset \mathscr{T}^*(p, \alpha_2, \beta_1).$$

Theorem 3 is an immediate conseqnence of the definition of the class $\mathscr{T}^*(p, \alpha, \beta)$.

THEOREM 4. *Let $-1 \leqq \alpha_1 \leqq \alpha_2 \leqq 1$, $0 \leqq \beta_1 \leqq \beta_2 \leqq 1$, and $\alpha_2 < \beta_1$. Then*

$$(3.2)\quad \mathscr{C}(p, \alpha_1, \beta_2) \supset \mathscr{C}(p, \alpha_2, \beta_1).$$

Proof. Let the function $f(z)$ defined by (1.1) be in the class $\mathscr{C}(p, \alpha_2, \beta_1)$ and $\beta_2 = \beta_1 + \varepsilon$. Then, by using Theorem 1,

$$(3.3)\quad \sum_{n=1}^{\infty}(p+n)^2(1+\beta_1)a_{p+n} \leqq (\beta_1-\alpha_2)p^2.$$

Hence

$$(3.4)\qquad \begin{aligned}&\sum_{n=1}^{\infty}(p+n)^2(1+\beta_2)a_{p+n}\\ &=\sum_{n=1}^{\infty}(p+n)^2(1+\beta_1+\varepsilon)a_{p+n}\\ &=\sum_{n=1}^{\infty}(p+n)^2(1+\beta_1)a_{p+n}+\varepsilon\sum_{n=1}^{\infty}(p+n)^2a_{p+n}\\ &\leqq(\beta_1-\alpha_2)p^2+\frac{\varepsilon(\beta_1-\alpha_2)p^2}{1+\beta_1}\\ &\leqq(\beta_1-\alpha_2)p^2+\varepsilon(\beta_1-\alpha_2)p^2\\ &\leqq(\beta_2-\alpha_1)p^2,\end{aligned}$$

which exhibits the fact that $f(z)\in\mathscr{C}(p,\ \alpha_1,\ \beta_2)$.

THEOREM 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{C}(p,\alpha,\beta)$. *Then* $f(z)$ *belongs to the class*

$$\mathscr{T}^*\left(p,\ \frac{p\beta+\alpha}{p+1},\ \beta\right),$$

that is,

$$(3.5)\qquad \mathscr{C}(p,\ \alpha,\ \beta)\subset\mathscr{T}^*\left(p,\ \frac{p\beta+\alpha}{p+1},\ \beta\right),$$

Proof. Assume that $f(z)\in\mathscr{C}(p,\ \alpha,\ \beta)$. Then, by means of Theorem 1, we have

$$(3.6)\qquad \begin{aligned}\sum_{n=1}^{\infty}(p+n)(1+\beta)a_{p+n}&\leqq\frac{(\beta-\alpha)p^2}{p+1}\\ &=\left(\beta-\frac{p\beta+\alpha}{p+1}\right)p^2.\end{aligned}$$

Note also that

$$-1\leqq\frac{p\beta+\alpha}{p+1}<\beta \text{ for } -1\leqq\alpha<\beta\leqq1,\ 0<\beta\leqq1,$$

and for $p\in\mathscr{N}$, and the assertion (3.5) follows at once.

4. Closure Theorems

Let the functions $f_i(z)$ be defined, for $i=1,\cdots,\ m$, by

$$(4.1)\qquad f_i(z)=z^p-\sum_{n=1}^{\infty}a_{i,\,p+n}z^{p+n}\quad(a_{i,\,p+n}\geqq0;\ p\in\mathscr{N})$$

for $z\in\mathscr{U}$.

We shall prove the following results for the closure of function in $\mathscr{T}^*(p,\ \alpha,\beta)$ and $\mathscr{C}(p,\ \alpha,\ \beta)$.

THEOREM 6. *Let the function* $f_i(z)$ *defined by* (4.1) *be in the class* $\mathscr{T}^*(p,\ \alpha_i,\ \beta_i)$ *for each* $i=1,\cdots,\ m$. *Then the function* $h(z)$ *defined by*

(4.2) $$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}a_{i,p+n}\right)z^{p+n}$$

is in the class $\mathcal{T}^*(p,\alpha,\beta)$, *where*

(4.3) $$\alpha=\min_{1\leq i\leq m}\{\alpha_i\} \quad \text{and} \quad \beta=\max_{1\leq i\leq m}\{\beta_i\}.$$

Proof. Since $f_i(z)\in\mathcal{T}^*(p,\alpha_i,\beta_i)$ for each $i=1,\cdots, m$, we observe that

(4.4) $$\sum_{n=1}^{\infty}(p+n)(1+\beta_i)\ a_{i,p+n}\leqq(\beta_i-\alpha_i)p$$

with the aid of Lemma 1.
Therefore

(4.5) $$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,p+n}\right)\\&=\frac{1}{m}\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{m}\left(\frac{p+n}{p}\right)a_{i,p+n}\right\}\\&\leqq\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\beta_i-\alpha_i}{1+\beta_i}\right)\\&\leqq\frac{\beta-\alpha}{1+\beta}.\end{aligned}$$

Thus

(4.6) $$\sum_{n=1}^{\infty}(p+n)(1+\beta)\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,p+n}\right)\leqq(\beta-\alpha)p,$$

which shows that $h(z)\in\mathcal{T}^*(p,\alpha,\beta)$, completing the proof of Theorem 6.

THEOREM 7. *Let the function* $f_i(z)$ *defined by* (4.1) *be in the class* $\mathcal{C}(p,\alpha_i,\beta_i)$ *for each* $i=1,\cdots, m$. *Then the function* $h(z)$ *defined by* (4.2) *is in the class* $\mathcal{C}(p,\alpha,\beta)$, *where* α *and* β *are defined by* (4.3)

The proof of Theorem 7 using Theorem 1 runs parallel to that of Theorem 6 using Lemma 1.

THEOREM 8. *Let the function* $f(z)$ *defined by* (1.1) *and the function* $g(z)$ *defined by*

(4.7) $$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n} \qquad (b_{p+n}\geqq 0;\ p\in\mathcal{N})$$

be in the classes $\mathcal{T}^*(p,\alpha,\beta)$ *and* $\mathcal{C}(p,\alpha,\beta)$, *respectively. Then the function* $k(z)$ *defined by*

(4.8) $$k(z)=z^p-\left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}(a_{p+n}+b_{p+n})z^{p+n}$$

is in the class $\mathcal{T}^*(p,\alpha,\beta)$.

Proof. Since

$$f(z)\in\mathcal{T}^*(p,\alpha,\beta) \text{ and } g(z)\in\mathcal{C}(p,\alpha,\beta),$$

we find from Lemma 1 and Theorem 1 that

(4.9) $$\sum_{n=1}^{\infty} (p+n)(1+\beta) a_{p+n} \leqq (\beta-\alpha) p$$

and

(4.10) $$\sum_{n=1}^{\infty} (p+n)(1+\beta) b_{p+n} \leqq \frac{(\beta-\alpha)p^2}{p+1}$$

Consequently, we have

(4.11) $$\left(\frac{p+1}{2p+1}\right) \sum_{n=1}^{\infty} (p+n)(1+\beta)(a_{p+n}+b_{p+n}) \leqq (\beta-\alpha) p$$

which implies that $k(z) \in \mathscr{T}^*(p, \alpha, \beta)$, and the proof of Theorem 8 is thus completed.

5. Extremal Properties of the Class $\mathscr{C}(p, \alpha, \beta)$

Shukla and Dashrath [9] proved the convexity of the class $\mathscr{T}^*(p, \alpha, \beta)$. We state the convexity of the class $\mathscr{C}(p, \alpha, \beta)$ as

THEOREM 9. *The class $\mathscr{C}(p, \alpha, \beta)$ is convex.*

Proof. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (4.7) be in the class $\mathscr{C}(p, \alpha, \beta)$. Then it is sufficient to prove that the function

$$r\, f(z) + (1-r)\, g(z) \quad (0 \leqq r \leqq 1)$$

is in the class $\mathscr{C}(p, \alpha, \beta)$. Since, for $0 \leqq r \leqq 1$,

(5.1) $$r\, f(z) + (1-r) g(z) = z^p - \sum_{n=1}^{\infty} \{r\, a_{p+n} + (1-r) b_{p+n}\} z^{p+n},$$

we readily have

(5.2) $$\sum_{n=1}^{\infty} (p+n)(1+\beta)\{r\, a_{p+n} + (1-r) b_{p+n}\}$$
$$= r \sum_{n=1}^{\infty} (p+n)^2 (1+\beta) a_{p+n} + (1-r) \sum_{n=1}^{\infty} (p+n)^2 (1+\beta) b_{p+n}$$
$$\leqq (\beta-\alpha) p^2,$$

by means of Theorem 1. Hence $\mathscr{C}(p, \alpha, \beta)$ is convex.

As a consequence of Theorem 9, there exists the extreme points of the class $\mathscr{C}(p, \alpha, \beta)$.

THEOREM 10. *Let*

(5.3) $$f_p(z) = z^p \qquad (p \in \mathscr{N})$$

and

(5.4) $$f_{p+n}(z) = z^p - \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} z^{p+n} \qquad (p \in \mathscr{N};\ n \in \mathscr{N}).$$

Then $f(z)$ belongs to the class $\mathscr{C}(p, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty} r_{p+n} f_{p+n}(z), \tag{5.5}$$

where

$$r_{p+n} \geqq 0 \text{ and } \sum_{n=0}^{\infty} r_{p+n}=1. \tag{5.6}$$

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} r_{p+n} f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} r_{p+n} z^{p+n}. \end{aligned} \tag{5.7}$$

Then it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ \frac{(p+n)^2(1+\beta)}{(\beta-\alpha)p^2} \cdot \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} r_{p+n} \right\} \\ &= \sum_{n=1}^{\infty} r_{p+n} = 1 - r_p \leqq 1, \end{aligned} \tag{5.8}$$

where we have also used the conditions in (5.6).

By virtue of Theorem 1, this shows that $f(z)$ belongs to the class $\mathcal{C}(p, \alpha, \beta)$.

On the other hand, suppose that the function $f(z)$ defined by (1.1) is in the class $\mathcal{C}(p, \alpha, \beta)$. Then, since

$$a_{p+n} \leqq \frac{(\beta-\alpha)p^2}{(p+n)^2(1+\beta)} \qquad (n \in \mathcal{N}), \tag{5.9}$$

we may put

$$r_{p+n} = \frac{(p+n)^2(1+\beta)}{(\beta-\alpha)p^2} a_{p+n} \qquad (n \in \mathcal{N}) \tag{5.10}$$

and

$$r_p = 1 - \sum_{n=1}^{\infty} r_{p+n}. \tag{5.11}$$

Consequently, we have the expression (5.5). This evidently completes the proof of Theorem 10.

COROLLARY 3. *The extreme points of the class* $\mathcal{C}(p, \alpha, \beta)$ *are the functions* $f_{p+n}(z)$ $(n \in \mathcal{N} \cup \{0\})$ *given by Theorem* 10.

6. Modified Hadamard Products

Let the function $f(z)$ be defined by (1.1) and the function $g(z)$ be defined by (4.7). Define the modified Hadamard product of two functions $f(z)$ and $g(z)$ by

$$f*g(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n} \tag{6.1}$$

We prove several interesting results involving the modified Hadamard product (6.1); our results are contained in Theorems 11 to 14 below.

THEOREM 11. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (4.7) be in the classes $\mathcal{T}^*(p, \alpha_1, \beta_1)$ and $\mathcal{T}^*(p, \alpha_2, \beta_2)$, respectively. Then the modified Hadamard product $f*g(z)$ defined by (6.1) belongs to the class*

$$\mathcal{T}^*(p, \ \alpha(2\beta-\alpha), \ \beta^2),$$

where

$$\alpha=\min\{\alpha_1, \alpha_2\} \text{ and } \beta=\max\{\beta_1, \beta_2\}. \tag{6.2}$$

Proof. In view of Lemma 1, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} (p+n)(1+\beta^2)\, a_{p+n}b_{p+n} &\leqq \sum_{n=1}^{\infty} (p+n)(1+\beta)\, a_{p+n}b_{p+n} \\ &\leqq (\beta-\alpha)p \cdot \frac{(\beta_0-\alpha)p}{(p+1)(1+\beta_0)} \\ &\leqq (\beta-\alpha)^2 p=[\beta^2-\alpha(2\beta-\alpha)]p,\end{aligned} \tag{6.3}$$

where β is given by (6.2), and (for convenience) $\beta_0=\min\{\beta_1, \beta_2\}$.

Now observe that

$$-1\leqq\alpha(2\beta-\alpha)<\beta^2\leqq 1 \text{ and } 0<\beta^2\leqq 1$$

for

$$-1\leqq\alpha<\beta\leqq 1 \text{ and } 0<\beta\leqq 1.$$

Hence

$$f*g(z)\in\mathcal{T}^*(p, \ \alpha(2\beta-\alpha), \ \beta^2).$$

In a similar manner we can prove

THEOREM 12. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (4.7) be in the classes $\mathcal{C}(p, \alpha_1, \beta_1)$ and $\mathcal{C}(p, \alpha_2, \beta_2)$, respectively. Then the modified Hadamard product $f*g(z)$ defined by (6.1) belongs to the class*

$$\mathcal{C}(p, \ \alpha(2\beta-\alpha), \ \beta^2),$$

where α and β are given by (6.2).

THEOREM 13. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (4.7) be in the same class $\mathcal{T}^*(p, \alpha, \beta)$. Then the modified Hadamard product $f*g(z)$ defined by (6.1) belongs to the class*

$$\mathcal{C}(p, \ \alpha(2\beta-\alpha), \ \beta^2).$$

Proof. Since

$$f(z)\in\mathcal{T}^*(p, \ \alpha, \ \beta) \text{ and } g(z)\in\mathcal{T}^*(p, \ \alpha, \ \beta),$$

we find from Lemma 1 that

$$\sum_{n=1}^{\infty} (p+n)^2(1+\beta^2)\, a_{p+n}b_{p+n} \tag{6.4}$$

$$\leqq \sum_{n=1}^{\infty} (p+n)^2(1+\beta)^2 a_{p+n} b_{p+n}$$
$$\leqq (\beta-\alpha)^2 p^2 = [\beta^2 - \alpha(2\beta-\alpha)] p^2.$$

Note also that

$$-1 \leqq \alpha(2\beta-\alpha) < \beta^2 \leqq 1 \text{ and } 0 < \beta^2 \leqq 1$$

for

$$-1 \leqq \alpha < \beta \leqq 1 \text{ and } 0 < \beta \leqq 1,$$

and Theorem 13 follows immediately.

THEOREM 14. *Let the function $f(z)$ defined by* (1.1) *and the function $g(z)$ defined by* (4.7) *be in the same class $\mathscr{C}^*(p, \alpha, \beta)$. Then the modified Hadamard product $f * g(z)$ defined by* (6.1) *belongs to the class*

$$\mathscr{C}\left(p, \frac{(1+2\alpha)\beta-\alpha^2}{1+\beta}, \beta\right).$$

Proof. It follows from Lemma 1 that

$$\sum_{n=1}^{\infty} (p+n)^2(1+\beta) a_{p+n} b_{p+n} \leqq \frac{(\beta-\alpha)^2 p^2}{1+\beta} \tag{6.5}$$
$$= \left[\beta - \frac{(1+2\alpha)\beta-\alpha^2}{1+\beta}\right] p^2.$$

Observe also that

$$-1 \leqq \frac{(1+2\alpha)\beta-\alpha^2}{1+\beta} < \beta \leqq 1$$

for

$$-1 \leqq \alpha < \beta \leqq 1 \text{ and } 0 < \beta \leqq 1,$$

and the proof of Theorem 14 is completed.

7. Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g., [1, Chapter 13], [4], [5], [7], [8], [10, p. 28 et seq.], and [12]). We find it to be convenient to recall here the following definitions which were used recently by Owa [6] (and by Srivastava and Owa [11]).

DEFINITION 1. *The fractional integral of order λ is defined, for a function $f(z)$, by*

$$D^{-\lambda}_z = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\lambda}}, \tag{7.1}$$

where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

DEFINITION 2. *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$(7.2) \qquad D^{\lambda}_z f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{\lambda}},$$

where $0\leqq\lambda<1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1 above.

DEFINITION 3. *Under the hypotheses of Definition 2, the fractional derivative of $f(z)$ of order $n+\lambda$ is defined by*

$$(7.3) \qquad D^{n+\lambda}_z f(z)=\frac{d^n}{dz^n}D^{\lambda}_z f(z),$$

where $0\leqq\lambda<1$ and $n\in\mathscr{N}\cup\{0\}$.

We need the following distortion theorem (given by Shukla and Dashrath [9]) for functions $f(z)$ belonging to the class $\mathscr{T}^*(p, \alpha, \beta)$.

LEMMA 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathscr{T}^*(p, \alpha, \beta)$. Then*

$$(7.4) \qquad |f(z)|\geqq|z|^p-\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|^{p+1},$$

$$(7.5) \qquad |f(z)|\leqq|z|^p+\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|^{p+1},$$

$$(7.6) \qquad |f'(z)|\geqq p|z|^{p-1}-\frac{(\beta-\alpha)p}{1+\beta}|z|^p,$$

and

$$(7.7) \qquad |f'(z)|\leqq p|z|^{p-1}+\frac{(\beta-\alpha)p}{1+\beta}|z|^p$$

for $z\in\mathscr{U}$. The estimates (7.4) *to* (7.7) *are sharp.*

We now state and prove

THEOREM 15. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathscr{T}^*(p, \alpha, \beta)$. Then*

$$(7.8) \qquad \left|D_z^{-\lambda}f(z)\right|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1-\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|\right\}$$

and

$$(7.9) \qquad \left|D_z^{-\lambda}f(z)\right|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1+\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|\right\}$$

for $\lambda>0$ and $z\in\mathscr{U}$. Furthermore

$$(7.10) \qquad \left|D_z^{1-\lambda}f(z)\right|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda-1}\left\{(p-\lambda)-\frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

and

$$(7.11)\quad \left|D_z^{1-\lambda} f(z)\right| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)} |z|^{p+\lambda-1}\left\{(p+\lambda)+\frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

for $\lambda>0$ *and* $z\in\mathcal{U}$. *The bounds* (7.8), (7.9), *and* (7.11) *are sharp.*

Proof. We begin by considering the function

$$(7.12)\quad \begin{aligned} F(z) &= \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} A_{p+n} z^{p+n}, \end{aligned}$$

where

$$(7.13)\quad A_{p+n} = \frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)} a_{p+n}.$$

Then, since

$$(7.14)\quad 0 < \frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1+\lambda)\Gamma(p+1)} < 1$$

for $\lambda>0, p\in\mathcal{N}$, and $n\in\mathcal{N}$, we have

$$0 < A_{p+n} < a_{p+n}.$$

It follows that

$$(7.15)\quad \begin{aligned} \sum_{n=1}^{\infty} (p+n)(1+\beta) A_{p+n} &\leqq \sum_{n=1}^{\infty} (p+n)(1+\beta) a_{p+n} \\ &\leqq (\beta-\alpha)p, \end{aligned}$$

which implies that $F(z)\in\mathcal{T}^*(p, \alpha, \beta)$. Consequently, by Lemma 2,

$$(7.16)\quad |F(z)| \geqq |z|^p - \frac{(\beta-\alpha)p}{(p+1)(1+\beta)} |z|^{p+1}$$

and

$$(7.17)\quad |F(z)| \leqq |z|^p + \frac{(\beta-\alpha)p}{(p+1)(1+\beta)} |z|^{p+1},$$

which, in view of the definition (7.12), are the assertions (7.8) and (7.9) of Theorem 15.

Next, by employing the second half of Lemma 2, we have

$$(7.18)\quad |F'(z)| \geqq p|z|^{p-1} - \frac{(\beta-\alpha)p}{1+\beta} |z|^p$$

and

$$(7.19)\quad |F'(z)| \leqq p|z|^{p-1} + \frac{(\beta-\alpha)p}{1+\beta} |z|^p$$

for $z\in\mathcal{U}$. Thus we obtain

$$(7.20)\quad \left|D_z^{1-\lambda} f(z)\right| \geqq \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)} |z|^{\lambda}\left\{p|z|^{p-1} - \frac{(\beta-\alpha)p}{1+\beta}|z|^p\right\}$$

$$-\lambda|z|^{-1}\left|D_z^{-\lambda}f(z)\right|$$
$$\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda+1}\left\{(p-\lambda)-\frac{(\beta-\alpha)(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

and

$$\left|D_z^{1-\lambda}f(z)\right|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{\lambda}\left\{p|z|^{p-1}+\frac{(\beta-\alpha)p}{1+\beta}|z|^{p}\right\}\tag{7.21}$$
$$+\lambda|z|^{-1}\left|D_z^{-\lambda}f(z)\right|$$
$$\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda-1}\left\{(p+\lambda)+\frac{(\beta-\alpha)(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

for $\lambda>0$ and $z\in\mathcal{U}$.

Finally, the bounds (7.8), (7.9), and (7.11) are sharp and are attained by the function $f(z)$ defined by

$$D_z^{-\lambda}f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}z^{p+\lambda}\left\{1-\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}z\right\}.\tag{7.22}$$

Similar are the proofs of our assertions (7.23) to (7.26) below.

THEOREM 16. *Let the function $f(z)$ defined by* (1.1) *be in the class* $\mathscr{C}(p, \alpha, \beta)$. *Then*

$$\left|D_z^{-\lambda}f(z)\right|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1-\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|\right\}\tag{7.23}$$

and

$$\left|D_z^{-\lambda}f(z)\right|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda}\left\{1+\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}|z|\right\}\tag{7.24}$$

for $\lambda>0$ and $z\in\mathcal{U}$ Furthermore

$$\left|D_z^{1-\lambda}f(z)\right|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda+1}\left\{(p-\lambda)-\frac{(\beta-\alpha)p^2(p+1+\lambda)}{(p+1)^2(1+\beta)}|z|\right\}\tag{7.25}$$

and

$$\left|D_z^{1-\lambda}f(z)\right|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}|z|^{p+\lambda+1}\left\{(p+\lambda)+\frac{(\beta-\alpha)p^2(p+1+\lambda)}{(p+1)^2(1+\beta)}|z|\right\}\tag{7.26}$$

for $\lambda>0$ and $z\in\mathcal{U}$. The bounds (7.23), (7.24), *and* (7.26) *are sharp and are attained by the function $f(z)$ defined by*

$$D_z^{-\lambda}f(z)=\frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)}z^{p+\lambda}\left\{1-\frac{(\beta-\alpha)p^2}{(p+1)^2(1+\beta)}z\right\}\tag{7.27}$$

Lastly, we prove

THEOREM 17. *Let the functionf(z) defined by* (1.1) *be in the class* $\mathscr{C}$ (p, α, β). *Then*

$$\left|D_z^{\lambda}f(z)\right|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-\lambda}\left\{1-\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|\right\}\tag{7.28}$$

and

$$(7.29)\qquad \left|D_z^{\lambda} f(z)\right| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-\lambda}\left\{1+\frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|\right\}$$

for $0 \leqq \lambda < 1$ *and* $z \in \mathcal{U}$. *Furthermore*

$$(7.30)\qquad \left|D_z^{1+\lambda} f(z)\right| \geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-1-\lambda}\left\{(p-\lambda)-\frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

and

$$(7.31)\qquad \left|D_z^{1+\lambda} f(z)\right| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-1-\lambda}\left\{(p+\lambda)+\frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

for $0 \leqq \lambda < 1$ *and* $z \in \mathcal{E}$, *where* $\mathcal{E}=\mathcal{U}$ *if* $p \geqq 2$, *and* $\mathcal{E}=\mathcal{U}-\{0\}$ *if* $p=1$. *The bounds* (7.28), (7.29), *and* (7.31) *are sharp.*

Proof. Define a function $G(z)$ by

$$(7.32)\qquad \begin{aligned} G(z) &= \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+1+\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} B_{p+n} z^{p+n}, \end{aligned}$$

where

$$(7.33)\qquad B_{p+n} = \frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)} a_{p+n}.$$

Since

$$(7.34)\qquad 1 \leqq \frac{\Gamma(p+n+1)\Gamma(p+1-\lambda)}{\Gamma(p+n+1-\lambda)\Gamma(p+1)} < \frac{p+n}{p}$$

for $0 \leqq \lambda < 1$, $p \in \mathcal{N}$, and $n \in \mathcal{N}$, it is readily seen that

$$(7.35)\qquad \begin{aligned} \sum_{n=1}^{\infty} (p+n)(1+\beta) B_{p+n} &\leqq \sum_{n=1}^{\infty} \frac{(p+n)^2}{p}(1+\beta) a_{p+n} \\ &\leqq (\beta-\alpha)p. \end{aligned}$$

Consequently, $G(z)$ is in the class $\mathcal{T}^*(p, \alpha, \beta)$, and we have

$$(7.36)\qquad |G(z)| \geqq |z|^p - \frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|^{p+1}$$

and

$$(7.37)\qquad |G(z)| \leqq |z|^p + \frac{(\beta-\alpha)p}{(p+1)(1+\beta)}|z|^{p+1},$$

which imply the first half of the theorem.

Now, from the second half of Lemma 2, we have

$$(7.38)\qquad |G'(z)| \geqq p|z|^{p-1} - \frac{(\beta-\alpha)p}{1+\beta}|z|^p$$

and

(7.39) $$|G'(z)| \leqq p|z|^{p-1} + \frac{(\beta-\alpha)p}{1+\beta}|z|^p$$

for $z \in \mathcal{U}$. Hence

(7.40) $$\left|D_z^{1+\lambda} f(z)\right| \geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{-\lambda}\left\{p|z|^{p-1} - \frac{(\beta-\alpha)p}{1+\beta}|z|^p\right\} - \lambda|z|^{-1}\left|D_z^{\lambda} f(z)\right|$$
$$\geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-1-\lambda}\left\{(\mathrm{p}-\lambda) - \frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

and

(7.41) $$\left|D_z^{1+\lambda} f(z)\right| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{-\lambda}\left\{p|z|^{p-1} + \frac{(\beta-\alpha)p}{1+\beta}|z|\right\} + \lambda|z|^{-1}\left|D_z^{\lambda} f(z)\right|$$
$$\leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}|z|^{p-1-\lambda}\left\{(p+\lambda) + \frac{(\beta-\alpha)p(p+1+\lambda)}{(p+1)(1+\beta)}|z|\right\}$$

for $0 \leqq \lambda < 1$ and $z \in \mathcal{E}$, where $\mathcal{E} = \mathcal{U}$ if $p \geqq 2$, and $\mathcal{E} = \mathcal{U} - \{0\}$ if $p=1$.

Finally, the bounds (7.28), (7.29), and (7.30) are sharp for the function $f(z)$ defined by

(7.42) $$D_z^{\lambda} f(z) = z^p - \frac{(\beta-\alpha)p}{(p+1)(1+\beta)} z^{p+1}.$$

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Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

Department of Mathematics
University of Victoria
Victorta, British Columbia V8W 2Y2
Canada