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MAXIMAL IDEALS IN POLYNOMIAL RINGS

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1. Introduction

Let R be a commutative noetherian ring with $1 \neq 0$, denoting by $\nu(I)$ the cardinality of a minimal basis of the ideal I. Let A be a polynomial ring in $n>0$ variables with coefficients in R, and let M be a maximal ideal of A. Generally it is shown that $\nu(MA_M) \leq \nu(M) \leq \nu(MA_M)+1$. It is well known that the lower bound is not always satisfied, and the most classical examples occur in nonfactional Dedekind domains. But in many cases, (e.g., A is a polynomial ring whose coefficient ring is a field) the lower bound is attained.

In [2] and [3], the conditions when the lower bound is satisfied is investigated. Especially in [3], it is shown that $\nu(M)=\nu(MA_M)$ if $M \cap R=p$ is a maximal ideal or A_M (equivalently R_P) is not regular or $n>1$. Hence the problem of determining whether $\nu(M)=\nu(MA_M)$ can be studied when p is not maximal, A_M is regular and $n=1$. The purpose of this note is to provide some conditions in which the lower bound is satisfied, when $n=1$ and R is a regular local ring (hence A_M is regular).

2. Cases when lower bound is satisfied

Let R be a regular local ring with the maximal ideal m in this section. Then we readily have the following Theorem.

THEOREM 2.1. *If M is a maximal ideal in $A=R[X]$ and $p=M \cap R$, and R/p is regular, then $\nu(M)=\nu(MA_M)$. Hence in this case, M is an ideal theoretic complete intersection.*

Proof. Since R/p is a regular local ring, R/p is a *Unique Factorization Domain*. But $M/p\ R[X] \subset A/pR[X] \simeq (R/p)[X]$. Now $M/p\ R[X]$ is an ideal of height 1 in U.F.D., $\nu(M/p\ R[X])=1$.

So $\nu(M) \leq \nu(p)+1 = htp+1$, since R/p is regular

$= \nu(p\ R_p)+1 = \nu(MA_M)$ (See Lemma 1, [3])

Therefore $\nu(M)=\nu(MA_M)$.

Now R_p is regular, equivalently A_M is regular, $\nu(MA_M)=ht(MA_M)=ht(M)$. So M is generated by a regular sequence. This means M is an ideal theoretic complete intersection.

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But we don't have enough evidence when R/p is singular.

In the case when R is a regular local ring of lower dimension it is shown that the lower bound holds for any maximal ideal M. For $\dim R=0$, p should be a maximal ideal, hence by Theorem 2 in [3], it is true. If $\dim R=1$ or $\dim R=2$, it is proved in [4] and [5] respectively. Now we will conditionally generalize this fact when $\dim R>2$. For this, we need the following basic theorems.

THEOREM 2.2. *If R is a noetherian ring and M a maximal ideal in $R[X_1, \cdots, X_n]$ and $M\cap R=p$, then $htM.=htp+n$.*

Proof. [6. Theorem 149].

THEOREM 2.3. *If R is a noetherian ring and $M\subset R[X_1, \cdots, X_n]$ is maximal and $p=M\cap R$ then R/p is a semilocal ring of dimension≤ 1.*

Proof. [1].

LEMMA 2.4. *If R is a regular local ring of dimension $d\geq 0$ and M is a maximal ideal in $A=R[X]$, then $ht\ M\geq d$.*

Proof. Set $M\cap R=p$. Then by Theorem 2.3, $\dim R/p\leq 1$. Since R is a regular local ring $\dim R-htp\leq 1$. Hence $htp=d$ or $d-1$. Now Lemma follows from Theorem 2.2.

Next, we come to state the main theorem.

THEOREM 2.5. *Suppose R is a regular local ring with the maximal ideal m and $\dim R=d\geq 0$. If M is a maximal ideal in $A=R[X]$ with a monic polynomial of degree greater than 1, then $\nu(M)=\nu(MA_M)$. Hence M is an ideal theoretic complete intersection.*

Proof. When $d\leq 2$, Theorem has been shown without the above extra condition. Hence we'll consider the case $d>2$, and prove the theorem by induction. If $htM=d+1$, then by Theorem 2.2, $htM\cap R=d$, hence $p=M\cap R$ is the maximal ideal. In this case by Theorem 2 in [3]theorem comes true. Therefore let's see the case when $htM=d$. First, we'll claim that there exists $\alpha\in p\backslash m^2$. For the claim choose a monic polynomial f in M of degree≥ 1. Consider $R[f]$. Then $R[X]$ is integral over $R[f]$. If we set $N=M\cap R[f]$, then N is a maximal ideal in $R[f]$ such that $N\cap R=M\cap R=p$. (Since $htp=d-1$, $p\neq m^2$.) Suppose $p\subsetneq m^2$. Then $N+m^2\ R[f]=R[f]$. Hence $1=(a_0+a_1f+\cdots+a_nf^n)+(b_0+b_1f+\cdots+b_sf^s)$. But $a_0+b_0=1$, $a_0\in N$ (since $f\in N$). Hence $a_0\in p$, $b_0\in m^2$, therefore $1\in m^2$, this is absurd. This means that $p\not\subset m^2$, and we can find $\alpha\in p\backslash m^2$. Let $\bar{R}=R/(\alpha)$, $\bar{M}=M/\alpha R[X]$. Then $\bar{R}$ is a regular local ring of dimension $d-1$ and $\bar{M}$ is a maximal ideal in $\bar{R}[X]$ satisfying the condition. By induction hypothesis $\nu(\bar{M})=\nu(\bar{M}\bar{R}[X]_{\bar{M}})$. Since $\bar{R}[X]$ is regular, $\nu(\bar{M})=ht(\bar{M})=d-1$ and therefore $\nu(M)=d$. Hence the proof is completed.

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