MAXIMAL IDEALS IN POLYNOMIAL RINGS

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1. Introduction

Let $R$ be a commutative noetherian ring with $1 \neq 0$, denoting by $\nu(I)$ the cardinality of a minimal basis of the ideal $I$. Let $A$ be a polynomial ring in $n>0$ variables with coefficients in $R$, and let $M$ be a maximal ideal of $A$. Generally it is shown that $\nu(MA_M) \leq \nu(M) \leq \nu(MA_M) + 1$. It is well known that the lower bound is not always satisfied, and the most classical examples occur in non-fractional Dedekind domains. But in many cases, (e.g., $A$ is a polynomial ring whose coefficient ring is a field) the lower bound is attained.

In [2] and [3], the conditions when the lower bound is satisfied is investigated. Especially in [3], it is shown that $\nu(M) = \nu(MA_M)$ if $M \cap R = p$ is a maximal ideal or $A_M$ (equivalently $R_\mathfrak{p}$) is not regular or $n \geq 1$. Hence the problem of determining whether $\nu(M) = \nu(MA_M)$ can be studied when $p$ is not maximal, $A_M$ is regular and $n = 1$. The purpose of this note is to provide some conditions in which the lower bound is satisfied, when $n = 1$ and $R$ is a regular local ring (hence $A_M$ is regular).

2. Cases when lower bound is satisfied

Let $R$ be a regular local ring with the maximal ideal $m$ in this section. Then we readily have the following Theorem.

**Theorem 2.1.** If $M$ is a maximal ideal in $A = R[X]$ and $p = M \cap R$, and $R/p$ is regular, then $\nu(M) = \nu(MA_M)$. Hence in this case, $M$ is an ideal theoretic complete intersection.

**Proof.** Since $R/p$ is a regular local ring, $R/p$ is a Unique Factorization Domain. But $M/p R[X] \subset A/p R[X] = (R/p)[X]$. Now $M/p R[X]$ is an ideal of height $1$ in U.F.D., $\nu(M/p R[X]) = 1$. So $\nu(M) \leq \nu(p) + 1 = htp + 1$, since $R/p$ is regular

$\nu(M) = \nu(p R_p) + 1 = \nu(MA_M)$ (See Lemma 1. [3])

Therefore $\nu(M) = \nu(MA_M)$.

Now $R_p$ is regular, equivalently $A_M$ is regular, $\nu(MA_M) = ht(MA_M) = ht(M)$. So $M$ is generated by a regular sequence. This means $M$ is an ideal theoretic complete intersection.

This work was supported by a grant from Ministry of Education, Korea in 1984.

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But we don’t have enough evidence when $R/p$ is singular.

In the case when $R$ is a regular local ring of lower dimension it is shown that the lower bound holds for any maximal ideal $M$. For $\dim R = 0$, $p$ should be a maximal ideal, hence by Theorem 2 in [3], it is true. If $\dim R = 1$ or $\dim R = 2$, it is proved in [4] and [5] respectively. Now we will conditionally generalize this fact when $\dim R > 2$. For this, we need the following basic theorems.

**Theorem 2.2.** If $R$ is a noetherian ring and $M$ a maximal ideal in $R[X_1, \ldots, X_n]$ and $M \cap R = p$, then $ht M = ht p + n$.

*Proof.* [6, Theorem 149].

**Theorem 2.3.** If $R$ is a noetherian ring and $M \subseteq R[X_1, \ldots, X_n]$ is maximal and $p = M \cap R$ then $R/p$ is a semilocal ring of dimension $\leq 1$.

*Proof.* [1].

**Lemma 2.4.** If $R$ is a regular local ring of dimension $d \geq 0$ and $M$ is a maximal ideal in $A = R[X]$, then $ht M \geq d$.

*Proof.* Set $M \cap R = p$. Then by Theorem 2.3, $\dim R/p \leq 1$. Since $R$ is a regular local ring $\dim R - htp \leq 1$. Hence $htp = d$ or $d - 1$. Now Lemma follows from Theorem 2.2.

Next, we come to state the main theorem.

**Theorem 2.6.** Suppose $R$ is a regular local ring with the maximal ideal $m$ and $\dim R = d \geq 0$. If $M$ is a maximal ideal in $A = R[X]$ with a monic polynomial of degree greater than 1, then $\nu(M) = \nu(MA M)$. Hence $M$ is an ideal theoretic complete intersection.

*Proof.* When $d \leq 2$, Theorem has been shown without the above extra condition. Hence we’ll consider the case $d > 2$, and prove the theorem by induction. If $htM = d + 1$, then by Theorem 2.2, $htM \cap R = d$, hence $p = M \cap R$ is the maximal ideal. In this case by Theorem 2 in [3] theorem comes true. Therefore let’s see the case when $htM = d$. First, we’ll claim that there exists $\alpha \in p \setminus m^2$. For the claim choose a monic polynomial $f$ in $M$ of degree $\geq 1$. Consider $R[f]$. Then $R[X]$ is integral over $R[f]$. If we set $N = M \cap R[f]$, then $N$ is a maximal ideal in $R[f]$ such that $N \cap R = M \cap R$. (Since $htp = d - 1$, $p \neq m^2$.) Suppose $p \subseteq m^2$. Then $N + m^2 R[f] = R[f]$. Hence $1 = (a_0 + a_1 f + \cdots + a_n f^n) + (b_0 + b_1 f + \cdots + b_j f^j)$. But $a_0 + b_0 = 1$, $a_0 \in N$ (since $f \in N$). Hence $a_0 \in p$, $b_0 \in m^2$, therefore $1 \in m^2$, this is absurd. This means that $p \subseteq m^2$, and we can find $\alpha \in p \setminus m^2$.

Let $\bar{R} = R/(\alpha)$, $\bar{M} = M/\alpha R[X]$. Then $\bar{R}$ is a regular local ring of dimension $d - 1$ and $\bar{M}$ is a maximal ideal in $\bar{R}[X]$ satisfying the condition. By induction hypothesis $\nu(\bar{M}) = \nu(\bar{M} R[X, \bar{r}])$. Since $\bar{R}[X]$ is regular, $\nu(\bar{M}) = h\bar{t}(\bar{M}) = d - 1$ and therefore $\nu(M) = d$. Hence the proof is completed.
References


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