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# A Relation between $\dim_A(E)=\dim_B(E)$ without the Finiteness on $E$*

by Chan-bong Park
*Wonkwang University, Iree, Korea*

## 1. Introduction

Throughout this note $B$ is a commutative ring with identity and $A$ is a subring of $B$ containing $1_B$. If every element of $B$ is integral over $A, B$ is said to be an integral extention of $A$.

Suppose that $B$ is an integral extention of $A$. When we regard $E$ as a finitely generated $A$-module and a finitely generated $B$-module, it has been already proved that their Krull dimensions are equal [6]. Even though $E$ has no finite condition will be they equal? The aim of this note is going to discuss above question. Notations and terminologies in this note are standard, and they are taken from [5] and [6]. The other notations will be discribed as it necessary and if there are those notations which were not mentioned previously it will be clear in the later discription. To get some corollaries by using the above result, we need the next lemma. The proof is routine and essentially well known, and will be omitted.

## 2. Lemmas and Results

**Lemma 1.** (i) *Let $\{B_i\}_{1\leq i\leq n}$ be a finite family of integral extentions of $A$ and let $B=\prod_{i=1}^{n}B_i$ be their product. Then $B$ is an integral extention of $A$.*

(ii) *Let $B$, $B'$* be extention rings of $A$. *If $B$ is an integral extention of $A$ then $B\bigotimes_A B'$ is an integral extention of $B'$.*

(iii) *Let $B$ be an integral extention of $A$ and $C$ is an integral extention of $B$. Then $C$ is an integral extention of $A$.*

(iv) *Let $B$ be an integral extension of $A$. Then $B[x]$ is an integral extension of $A[x]$, where $x$ is an indeterminate.*

**Lemma 2.** *Let $E$ be an $A$-module. Then the $A$-homomorphism*

$$E\xrightarrow{f}\prod_{\mathscr{P}\in\operatorname{Spec}(A)} E_{\mathscr{P}},$$

*induced by the canonical homomorphisms $E\to E_{\mathscr{P}}$ is injective. In particular, $E=0$ if and only if* $\operatorname{Supp}_A(E)=\{\mathscr{P}\mid\mathscr{P}\in\operatorname{Spec}(A),\ E_{\mathscr{P}}\neq 0\}=\phi$.

**Proof.** Let $x\in E$ be such that $f(x)=0$. This means that, for every $\mathscr{P}\in\operatorname{Spec}(A)$, there exists $s\in A-\mathscr{P}$ such that $sx=0$. Thus $\operatorname{ann}(Ax)\not\subset\mathscr{P}$, for every $\mathscr{P}\in\operatorname{Spec}(A)$, so that $\operatorname{ann}(Ax)=A$ and

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$x=0$.

**Theorem** *Let B be an integral extention of A and let E regard as an A-module, and a B-module. Then*

$$\dim_A(E)=\dim_B(E),$$

where $\dim_A(E)$ means the Krull dimension of $E$ over $A$.

**Proof.** By definition of the Krull dimension we may let

$$\dim_B(E)=\operatorname{Sup}\{\dim B/P \mid P\in \operatorname{Spec}(B),\ E_p\neq 0\}.$$

Let $n=\dim_B(E)$ and let $P\in\operatorname{Spec}(B)$ such that $E_p\neq 0$. Put $\mathscr{P}=P\cap A$. Then $B/P$ is an integral extention of $A/\mathscr{P}$, hence $\dim B/P=\dim A/\mathscr{P}$. Moreover $E_P$ is a localization of $E_{\mathscr{P}}$, therefore $E_{\mathscr{P}}\neq 0$. Thus $\dim_A(E)\geq\dim_B(E)$. To prove the converse let $\mathscr{P}\in\operatorname{Spec}(A)$ be such that $\dim A/\mathscr{P}=\dim_A(E)$, $E_{\mathscr{P}}\neq 0$. We have to prove that there exists $P\in\operatorname{Spec}(B)$ lying over $\mathscr{P}$ such that $E_P\neq 0$. Replaceing $A, B, E$ by $A_{\mathscr{P}}$, $B_{\mathscr{P}}$, $E_{\mathscr{P}}$ we may suppose that $(A,\mathscr{P})$ is a local ring and $E\neq 0$. Then the prime ideals of $B$ lying over $\mathscr{P}$ are exactly the maximal ideals of $B$, and hence by lemma 2 $\operatorname{Supp}_B(E)\neq 0$, and therefore

$$\dim_A(A)\leq\dim_B(E),$$

which is proved.

**Corollary** 1. *Let* $\{B_i\}_{1\leq i\leq n}$ *be a finite family of integral extentions of A and* $B=\prod_{i=1}^{n} B_i$ *be their product. Then*

$$\dim_A(E)=\dim_B(E).$$

**Proof.** It suffices to show that $B$ is an integral extension of $A$. This fact comes from lemma 1(i) and theorem.

**Corrollary** 2. *Let B, B′ be integral extentions of A. Then*

$$\dim_A(E)=\dim_{B\otimes_A B'}(E).$$

**Proof.** This fact comes from lemma 1(ii) with a slight modification, and Lemma 1(iii) and theorem.

**Corollary** 3. *Let B be an integral extention of A and C an integral extention of B. Then*

$$\dim_A(E)=\dim_C(E).$$

**Proof.** By Lemma 1(iii) and theorem.

**Corollary** 4. *Let B be an integral extension of A and E regard as both* $A[x]$ *and* $B[x]$*-module. Then*

$$\dim_{A[x]}(E)=\dim_{B[x]}(E).$$

**Proof.** By Lemma 1 (iv) and theorem.

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