韓國數學教育學會誌<數學教育>
1985. 6, Vol. XXIII, No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1985, Vol. XXIII, No. 2

An Elementary Proof of the Stone-Weierstrass Theorem

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The Stone-Weierstrass theorem is among the most important in modern abstract analysis. Many proofs of this important theorem have evolved since its discovery by M.H. Stone in 1937. In contrast to many proofs of the Stone-Weierstrass theorem Brosowski and Deutsch [1] have proved the theorem by using the elementary Bernoulli inequality: $(1+x)^n \geqslant 1+nx$, $n=1, 2, \ldots$ if $x \geqslant -1$. Using [1] we give an elementary proof of this theorem without appealing to the classical Weierstrass theorem (nor even the special case of uniformly approximating $P_n(x)=|x|$ on $[-1, 1]$ by polynomials) in this note.

Let X be a compact subspace of the Euclidean space $\boldsymbol{R}^n$ and $C(X)$ the set of all continuous real-valued functions on X. If addition and scalar multiplication are defined as usual on $C(X)$ by $(f+g)(x)=f(x)+g(x)$ and $(\alpha f)(x)=\alpha f(x)$ for all $f, g \in C(X)$ and $\alpha \in \boldsymbol{R}$, then $C(X)$ becomes a vector space. If we also define a product on $C(X)$ by $(fg)(x)=f(x)g(x)$ for all $f, g \in C(X)$, then $C(X)$ becomes an algebra with identity. Now let $\mathscr{A}$ be a subalgebra of $C(X)$, that is, $\mathscr{A}$ is a linear subspace of $C(X)$ that contains the product of each pair of its elements. We say that $\mathscr{A}$ ***separates points*** of X if, whenever x and y are distinct points of X, there is an element $f \in \mathscr{A}$ with $f(x) \neq f(y)$. $\mathscr{A}$ is said to be ***uniformly closed*** if whenever $f_n \in \mathscr{A}$ $(n=1, 2, \cdots)$ and $f_n \rightarrow f$ uniformly on X, then f is also in $\mathscr{A}$. If $\mathscr{A}$ is an arbitrary subalgebra of $C(X)$, the ***uniform closure*** of $\mathscr{A}$, denoted by $\bar{\mathscr{A}}$, is the set of all elements of $C(X)$ that are limits of uniformly convergent sequences of elements of $\mathscr{A}$; equivalently, $\bar{\mathscr{A}}$ is the set of all elements of $\mathscr{A}$ to any desired degree of accuracy. Clearly $\bar{\mathscr{A}}$ is also a subalgebra of $C(X)$.

The classical Weierstrass theorem says that if f is a continuous real-valued function on the closed interval $[a, b]$, then there exists a sequence of real polynomials P_n such that $\lim_{n\to\infty} P_n(x)=f(x)$ uniformly on $[a, b]$, that is, the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

The Stone-Weierstrass theorem may be stated as follows:

Theorem 1. *If $\mathscr{A}$ is a subalgebra of $C(X)$ which contains nonzero constant functions and separates points of X, then $\bar{\mathscr{A}}=C(X)$, that is, the elements of $C(X)$ can be uniformly approximated by the elements of $\mathscr{A}$. More precisely, given $f \in C(X)$ and $\varepsilon>0$, there exists $g \in \mathscr{A}$ such that $\sup_{x\in X}|f(x)-g(x)| < \varepsilon$, where the supremum norm $\|f\|=\sup_{x\in X} |f(x)|$ is given in $C(X)$.*

Proof. In [2] the proof is given by four steps and the classical Weierstrass theorem is used only in the first step. In fact, the first step is proved by using the following which is a special case of the classical Weierstrass theorem: For every interval $[-1, 1]$ there is a sequence of real polynomials P_n such that $P_o(0)=0$ and $\lim_{n\to\infty} P_n(x)=|x|$ uniformly on $[-1, 1]$. Thus it suffices to prove that the following lemmas which make it possible to replace the special case of the classical Weierstrass theorem. For detailed proof (from step 1 to step 4), see [2, pp. 162-164].

Lemma 2. *Let x and a be two real numbers with $0\leqslant x\leqslant a\leqslant 1$. Consider the sequence of real numbers defined by $x_0=0$ and $x_{n+1}=x_n+1/2(a^2-x_n^2)$ for $n=1, 2, \cdots$. Then $\lim_{n\to\infty} x_n=a$.*

Proof. Note that $0\leqslant x\leqslant x+1/2(a^2-x^2)\leqslant a$. For, $x^2-2x+2a-a^2$ is decreasing whenever $0\leqslant x\leqslant 1$ and $x^2-2x+2a-a^2=0$ whenever $x=a$. Thus from this fact we have $0\leqslant x_n\leqslant x_{n+1}\leqslant a$, $n=1, 2, \cdots$. Hence $\lim_{n\to\infty} x_n=x+1/2(a^2-x_n^2)$ as $n\to\infty$. This means that $\lim_{n\to\infty} x_n=a$.

Lemma 3. *For $-1\leqslant x\leqslant 1$, suppose the sequence of polynomials defined by $P_0(x)=0$ and $P_{n+1}(x)=P_n(x)+1/2[x^2-P_n^2(x)]$ for $n=1, 2, \cdots$. Then $\lim_{n\to\infty} P_n(x)=|x|$ uniformly on $[-1, 1]$.*

Proof. The given polynomials stem from the sequence in Lemma 2. Thus if we prove that $0\leqslant P_n(x)\leqslant P_{n+1}(x)\leqslant |x|$, that is, the sequence of polynomials is bounded and monotone increasing, then the result follows from Lemma 2 and the famous Dini's theorem (if $\lim_{n\to\infty} P_n(x)=|x|$ for $-1\leqslant x\leqslant 1$, then $\lim_{n\to\infty} P_n(x)=|x|$ unifromly on $[-1, 1]$).

Clearly, $0\leqslant P_n(x)\leqslant P_{n+1}(x)$. Thus it suffices to prove that $0\leqslant |x|-P_n(x)\leqslant |x|\left(1-\frac{|x|}{2}\right)^n$ for $-1\leqslant x\leqslant 1$ by induction. First, we have

$$\begin{aligned}|x|-P_{n+1}(x)&=|x|-P_n(x)-1/2[x^2-P_n^2(x)]\\&=|x|-P_n(x)-1/2[x^2-P_n^2(x)]-1/2|x|P_n(x)+1/2|x|P_n(x)\\&=[|x|-P_n(x)]\ [1-1/2(|x|+P_n(x))].\end{aligned}$$

When $n=0$, it is trivial. Suppose $n>0$. By induction hypothesis, $0\leqslant |x|-P_n(x)\leqslant |x|$. Thus we get $0\leqslant P_n(x)\leqslant |x|$. Hence $1/2[|x|+P_n(x)]\leqslant |x|\leqslant 1$. So we can derive that $|x|-P_{n+1}(x)\geqslant 0$. Thus $0\leqslant P_n(x)\leqslant P_{n+1}(x)\leqslant |x|$ for $-1\leqslant x\leqslant 1$. Consequently, we have

$$|x|-P_{n+1}(x)\leqslant |x|-P_n(x)\leqslant |x|\left(1-\frac{|x|}{2}\right)^n\left(1-\frac{|x|}{2}\right)=|x|\left(1-\frac{|x|}{2}\right)^{n+1}.$$

References

1. B. Brosowski and F. Deutsch, An elementary proof of the Stone-Weierstrass theorem, *Proc. Amer. Math. Soc.*, 81 (1981), 89-92.
2. W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, Inc., 1976.