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Hausdorff Measure on Some Metric Space*

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In many contributions to the theory of Hausdorff measure, it has been the practice to place certain restrictions on the spaces and the measure functions used. The purpose of this note is to find conditions under which the restrictions may be relaxed. In [4], it was shown that a σ-compact set E in a metric space has $\mu^h(E)=0$ for some $h\in H$.

We first show that a σ-precompact set E in a metric space has $\mu^h(E)=0$ for some h in H using the results which are derived from [1, 4].

Finally we will show that there can be uncountable set K which has $\mu^h(K)=0$ for all $h\in H_0$.

Throughout this note, X will denote a metric space with metric ρ. We use H to denote the class of function h defined for all $t\geq 0$, but perhaps having the value $+\infty$ for some values of t, monotonic increasing for $t\geq 0$, positive for $t>0$ and continuous on the right for all $t\geq 0$. We will use H_0 for the subset of all h of H with $h(0)=0$.

The Hausdorff h-measure $\mu^h(E)$ of a set E in a metric space X, for a function $h\in H$, is defined in the following way.

For each $\delta>0$ we let

$$\mu_\delta{}^h(E)=\inf_{\bigcup_{i=1}^{\infty}S_i\supset E}\sum_{i=1}^{\infty}h(S_i)$$

where the infimum being taken over all countable covering of E by open sets of diameter less than or equal to δ, and where

$h(S_i)=h(\mathrm{diam}(S_i))$, $\mathrm{diam}(S_i)$ being the diameter of S_i. Then $\mu^h(E)=\lim_{\delta\to 0}\mu_\delta{}^h(E)$ is an outer measure.

A metric space X is said to be precompact [5] if every sequence of points in X contains a Cauchy sequence, and X is said to be σ-precompact if X is the countable union of precompact sets. If X is complete precompact space, X is necessarily compact.

Lemma 1. ([5]). *A subset M of a metric space X is precompact (in the metric ρ of X) if and only if, given any $\varepsilon>0$, X contains a finite set $B\subset X$ such that the distance from every point $x\in M$ to some point $y\in B$ (in general, depending on ε) does not exceed ε.*

In fact, M is precompact iff for every $\varepsilon>0$, it can be covered by a finite number of open balls $O_\varepsilon(x_1),\cdots,O_\varepsilon(x_n)$ with radius ε.

Theorem 2. *The followings are equivalent*

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(1) *A set E has zero h-measure.*

(2) *There is a sequence* $\langle E_i\rangle$ *of sets,* $\sum_{i=1}^{\infty} h(E_i)$ *is finite so that each point of E belongs to infinitely many of the sets* E_i.

(3) *There is a sequence* $\langle E_i\rangle$ *of sets such that*

$$E\subset\bigcup_{j=i}^{\infty} E_j,\quad i=1,2,\cdots \textit{ and } \operatorname{diam}(E_i)\to 0,\quad \textit{as } i\to\infty.$$

Proof. (1)⇒(2) Let $\mu^h(E)=0$. By the definition of μ^h, we can choose a sequence of sets $\langle E_i^n\rangle$ for each $n\geq 1$ with

$$E\subset\bigcup_{i=1}^{\infty} E_i^{(n)},\quad \sum_{i=1}^{\infty} h(E_i^{(n)})<1/2^n.$$

From the rearranging the set E_i^n, i, $n=1,2,\cdots$ as a single sequence,

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} h(E_i^{(n)})<1$$

and each point of E belongs to infinitely many of the sets $E_i^{(n)}$.

(2)⇒(3) From the hypothesis, it is possible to choose a sequence $\langle E_i\rangle$ of sets such that

$$E\subset\bigcup_{j=i}^{\infty} E_j,\quad i=1,2,\cdots$$

and $\sum_{i=1}^{\infty} h(\operatorname{diam}(E_i))$ is finite.

Since $h(\delta)>0$ for any given $\delta>0$, $\operatorname{diam}(E_i)\geq\delta$ for at most finite number of values of i. Hence $\operatorname{diam}(E_i)\to 0$ as $i\to\infty$.

(3)⇒(1) Since $E\subset\bigcup_{j=i}^{\infty} E_j$, $i=1,2,\cdots$ and $\operatorname{diam}(E_i)\to 0$ as $i\to\infty$,

it is possible to choose a strictly increasing continuous Hausdorff measure function h such that

$$\sum_{i=1}^{\infty} h(\operatorname{diam}(E_i))<\infty.$$

Let N be so large that $\sum_{i=N}^{\infty} h(\operatorname{diam}(E_i))<\infty$, then

$$E\subset\bigcup_{j=N}^{\infty} E_j,\quad \operatorname{diam}(E_j)<\delta \text{ for } i\geq N,\quad \sum_{i=N}^{\infty} h(E_i)<\varepsilon.$$

Hence $\mu_\delta{}^h(E)<\varepsilon$. Therefore $\mu^h(E)=0$.

Lemma 3. *Any σ-precompact space X has a cover of sequence* $\langle U_i\rangle$ *of sets such that* $\operatorname{diam}(U_i)\to 0$ *as* $i\to\infty$ *and* $X\subset\bigcup_{j=i}^{\infty} U_j$, $i=1,2,\cdots$.

Proof. Let X be the countable union of precompact subset X_k of X. If, for every k, $\langle U_1^k, U_2^k, \cdots, U_n^k\rangle_{k\in N}$ is a countable cover of X by sets of diameter $<2^{-n}$. The diameter also converges to zero since only finitely many $\operatorname{diam}(U_i^k)$ are greater than a given $\varepsilon>0$.

Therefore, from the union of the above covers for $\varepsilon=1/n$, $n\in N$, we can find a required cover of equence $\langle U_i\rangle$.

Theorem 4. *Let E be a σ-precompact set in a metric space X. Then there exists an h in H for which $\mu^h(E)=0$.*

Proof. Let $E=\bigcup_{i=1}^{\infty} E_i$, where each set E_i is a precompact subset of E. Put

$$X_i=\bigcup_{j=1}^{i} E_i,\quad i=1,2,\cdots$$

hen each set X_i is precompact,

$$X_1\subset X_2\subset X_3\subset\cdots,\ \text{and}\ E=\bigcup_{i=1}^{\infty} X_i.$$

For each i, the system of open spheres

$$S(x,1/i)\ \text{for}\ x\in X_i$$

orms an open cover of the precompact set X_i with diameter $<1/i$.

Let

$$S(x_{ij},1/i),\quad j=1,2,\cdots,\ J_i$$

be a finite subcover of X_i by such spheres. Let $S_1, S_2, \cdots$ be an enumeration of these spheres in their natural order, taking first whose associated with X_1, then those associated with X_2, and so on. Then

$$\operatorname{diam}(S_k)<2/i\quad\text{if}\quad k>J_1+J_2+\cdots+J_{i-1}.$$

Now we can choose h to be a continuous and monotonic increasing function satisfying the conditions

$$h(0)=0,\ \ h(2/i)=(J_1+J_2+\cdots+J_i)^{-2},\ \ i=1,2,3,\cdots.$$

Now given any $k>J_1$, there will be an i with

$$J_1+J_2+\cdots+J_{i-1}<k\leq J_1+J_2+\cdots+J_i,$$

and with this value of i,

$$h(S_k)=h(\operatorname{diam}(S_k))\leq h(2/i)=(J_1+J_2+\cdots+J_i)^{-2}\leq h^{-2}.$$

Hence the series $\sum_{k=1}^{\infty} h(S_k)$ converges. But each point of E belongs to all sets X_i from some point onwards and so to infinitely many sets S_k.

Thus from the above Theorem 2 and Lemma 3, we can know that $\mu^h(E)=0$.

For an uncountable complete metric space, the following theorem was known in [4].

Theorem 5. *Let X be an uncountable complete separable metric space. Then there is a compact perfect subset E of X and a function h of H_0 with*

$$0<\mu^h(C)<+\infty.$$

It is easy to show that the theorem 5 holds even if the compact perfect set is replaced the precompact perfect set C. By the similar methods as in [4], the following theorems hold.

Theorem 6. *If X is a complete separable metric space and $\mu^h(K)=0$ for each h in H_0, and each precompact set K in X, then X is countable.*

Theorem 7. *If K is a precompact set in a metric space X and $\mu^h(K)=0$ for all h in H_0, then K is countable.*

By the theorem 5, for an uncountable compact set K in a metric space X, there is a perfect subset P of K and a function $h \in H_0$ such that $0 < \mu^h(P) < +\infty$.

Theorem 8. *Let K be an uncountable set in a metric space X and C be a countable set such that each open set containing C contains all points of K with at most a finite number of exceptions, then $\mu^h(K)=0$ for all $h \in H_0$.*

Proof. Let h be any function of H_0 and let $\varepsilon>0$, $\delta>0$ be given. Since $h(t)\to 0$ as $t\to 0$, we can choose a sequence $\langle r_1, r_2, \cdots\rangle$ of radii so that

$$2r_i<\delta \quad (i=1,2,\cdots), \quad \sum_{i=1}^{\infty} h(2r_i)<\varepsilon/2.$$

Then the sequence of open spheres

$$S(d_i, r_i), \quad i=1,2,\cdots, \quad d_i \in C$$

has a union covering C.

From the hypothesis, the points of K not covered by the union of open spheres can be enumerated as a sequence $\langle k_1, k_2, \cdots\rangle$, which might terminate or be empty.

New K is covered by the system of spheres

$$S(d_1, r_1),\ S(d_2, r_2),\ \cdots,\ S(k_1, r_1),\ S(k_2, r_2),\ \cdots,$$

all of diameter less than δ, with

$$h(S(d_1,r_1))+h(S(d_2,r_2))+\cdots+h(S(k_1,r_1))+h(S(k_2,r_2))+\cdots \leq 2\sum_{i=1}^{\infty} h(2r_i) < \varepsilon.$$

Hence $\mu_\delta{}^h(K)<\varepsilon$ for any $\varepsilon>0$.

Since $h\in H_0$, $\varepsilon>0$ and $\delta>0$ are arbitrary, $\mu^h(K)=0$.

References

1. C. Bandt, Spaces of largest Hausdorff dimension, *Mathematika*, 28 (1981), 206-210.
2. R.O. Davies, Increasing sequences of sets and Hausdorff measure, *Proc. London Math. Soc.* (3), 20 (1970), 222-236.
3. R.J. Gardner, Approximation sequences and Hausdorff measure, *Proc. Camb. Phil. Soc.*, (1974), 161-172.
4. C.A. Rogers, *Hausdorff measure*, Cambridge, 1970.
5. G.E. Shilov, *Elementary functional analysis*, MIT press, 1974.