(國數學教育學會誌<數學敎育>
)85. 6, Vol. XXIII, No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1985, Vol. XXIII, No. 2

# On Best Approximation in Metric Spaces

by T.D. Narang*

*Guru Nanak Dev University, Amritsar, India*

The problem of best approximation has been extensively studied in normed linear spaces (see e.g. 3], [4] and [5]). The same problem in metric spaces has been studied by a very few mathema- icians. The results available in such spaces do not constitute a unified theory. The author in a series f papers has made an attempt in this direction. The present paper is also a step in the same direc- ion. Theorem 1 generalizes a result of [1] on best simultaneous approximation, Theorem 2 gener- lizes a result of [6] on the continuity of metric projections and Theorem 3 generalizes a result iven in [4] on the Lipschitzian metric projections. To start with, we recall a few definitions.

Let $G$ be a non-empty subset of a metric space $(X, d)$.

An element $g_0 \in G$ is said to be a *best approximation* to an element $x \in X$ in $G$ if $d(x, g_0) = d(x, G)$ and it is said to be a *strongly unique element of best approximation* of $x$ in $G$ if there xists a constant $r = r(x, G)$ with $0 < r \leqslant 1$ such that $d(x, g) \geqslant d(x, g_0) + rd(g_0, g)$ for all $g \in G$.

The mapping which takes each point of $X$ to set of its best approximations in $G$ is called the *etric projection* of $X$ onto $G$.

The set $G$ is said to be:

(i) *Chebyshev* (*strongly Chebyshev*) if every point of $X$ has a unique best approximation (strongly inique element of best approximation) in $G$.

(ii) *P-compact* if for each $x \in X$, the set $P_G(x) = \{y \in G : d(x, y) = d(x, G)\}$ is non-empty and ompact.

(iii) *δ-compact* or *spherically compact* if for all $x \in G$ there exists a $\delta > 0$ such that the set $\{y \in G : d(x, y) \leqslant d(x, G) + \delta\}$ is compact.

(iv) *approximatively compact* if for every $x \in X$ and every sequence $\langle g_n \rangle$ in $G$ with $\lim_{n \to \infty} d(x, g_n) = d(x, G)$ there exists a subsequence $\langle g_{n_k} \rangle$ converging to an element of $G$.

(v) *locally compact* if for any $x \in G$ there exists an $r > 0$ such that the set $\{y \in G : d(x, y) \leqslant r\}$ s compact.

(vi) $\overset{\circ}{V}$*-connected* if for each open ball $\overset{\circ}{V}$, the set $G \cap \overset{\circ}{V}$ is empty or connected.

Let $X$ and $Y$ be two metric spaces. A mapping $f : X \to 2^Y$, the collection of all subsets of $Y$, is said to be *upper semi-continuous* if the set $\{x \in X : f(x) \in M\}$ is open for every open $M \subset Y$.

Let $C$ be an arbitrary subset of a metric space $(X, d)$ and $F$ be a bounded subset of $X$. An element

* The author is thankful to U.G.C., India for financial support.

$x^* \in C$ is said to be a *best simultaneous approximation* to the set $F$ if $\sup_{y\in F} d(y, x^*) = \inf_{x\in C} \sup_{y\in F} d(y, x)$.

The following theorem gives the existence of elements of best simultaneous approximation in metric spaces.

**Theorem 1.** *Let $C$ be a compact subset and $F$ be a bounded subset of a metric space $(X, d)$. Then there exist a best simultaneous approximation in $C$ to $F$.*

**Proof.** Consider the function $\phi : C \to \mathbf{R}$ defined by

$$\phi(x) = \sup_{y\in F} d(y, x).$$

This function is continuous on $C$. Since $C$ is compact, $\phi$ attains its infimum at some $x^* \in C$, i.e.,

$$\sup_{y\in F} d(y, x^*) = \phi(x^*) = \inf_{x\in C} \phi(x) = \inf_{x\in C} \sup_{y\in F} d(y, x).$$

**Note.** For normed linear spaces this result was proved in [1].

D.E. Wulbert proved in [6] that a locally compact, $\overset{\circ}{V}$-connected Chebyshev set in a Banach space is $\delta$-compact, approximatively compact and has a continuous metric projection. Wulbert's method extends to the following more general situation without essential changes.

**Theorem 2.** *In a metric space $(X, d)$ every locally compact, $P$-compact, $\overset{\circ}{V}$-connected set $G$ is $\delta$-compact, approximatively compact and its metric projection is upper semi-continuous.*

**Proof.** The proof of the $\delta$-compactness of $G$ is exactly similar to the corresponding proof given for normed linear spaces in [5]-Theorem 2.2. It is approximatively compact as every $\delta$-compact set in a metric space is approximatively compact (see [2]-Theorem 2). Since $G$ is approximatively compact, the metric projection is upper semi-continuous (Theorem 3.1 [4]-page 386).

Since for a Chebyshev set the metric projection is single-valued and for single-valued maps the two concepts of upper semicontinuity and continuity coincide, we have

**Corollary.** *In a metric space $(X, d)$ every locally compact, $\overset{\circ}{V}$-connected Chebyshev set is $\delta$-compact, approximatively compact and has a continuous metric projection.*

Finally, we discuss condition under which a metric projection is pointwise Lipschitzian.

**Theorem 3.** *For every strongly Chebyshev subset $G$ of a metric space $(X, d)$, the metric projection $\pi_G$ is pointwise Lipschitzian, i.e., for each $x \in X$, there exists a constant $\alpha = \alpha(x, G)$, such that*

$$d(\pi_G(x), \pi_G(y)) \leqslant \alpha d(x, y), \quad y \in X.$$

**Proof.** Since $G$ is strongly Chebyshev, there exists a unique $g_0 \in G$ and a constant $r = r(x, G)$ with $0 < r \leqslant 1$ such that

$$d(x, g) \geqslant d(x, g_0) + r d(g_0, g), \quad g \in G.$$

Putting $g_0 = \pi_G(x)$ and $g = \pi_G(y)$, we obtain

$$\begin{aligned} r d(\pi_G(\pi), \pi_G(y)) &\leqslant d(x, \pi_G(y)) - d(x, \pi_G(x)) \\ &\leqslant d(x, y) + d(y, \pi_G(y)) - d(x, \pi_G(x)) \\ &\leqslant d(x, y) + d(y, \pi_G(x)) - d(x, \pi_G(x)) \end{aligned}$$

$$\leqslant d(x,y)+d(y,x)+d(x,\pi_G(x))-d(x,\pi_G(x))$$
$$=2d(x,y)$$

ıd so taking $\alpha=\frac{2}{r}$, we get the result.

**Note.** For normed linear spaces this result was proved by G.Freud and E.W. Cheney (see [4]-age 49).

## References


. T.N. Narang, On best simultaneous approximation, *J. Approximation Theory*, 39(1983), 93-96.

. T.D. Narang and Swaran Khanna, Inter-relationship between sets useful in solving best approximation problems, *Math. Education*, 15(1981), 48-50.

. Ivan Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, New York, 1970.

. Ivan Singer, The theory of best approximation and functional analysis, *CBMS Regional Conference Series in Applied Math.*, 13(1974), Philadelphia, SIAM.

. L.P. Vlasov, Approximative properties of sets in normed linear spaces, *Russian Math. Surveys*, 28(1973), 1-66.

6. D.E. Wulbert, Continuity of metric projections, *Trans. Amer. Math. Soc.*, 134(1968), 335-341.