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On Some Common Fixed Point Theorems in Probabilistic Metric Spaces

by Sang-Tae Ro

Joong-Kyeong Technical Junior College, Daejoun, Korea

Abstract

In this paper we obtain several results on the existence of common fixed points of commuting appings on PM-spaces and give an application. Our results generalize a multitude of fixed point .eorems in PM-spaces.

1. Preliminaries

We first give several definitions and preliminary results for probabilistic metric spaces and its pological properties. For more detailed discussions of probabilistic metric spaces and their properties, we refer to [1∼5, 7, 9∼10].

Let R denote the real numbers and $R^+=\{x\in R;\ x\geq 0\}$.

Definition 1.1. A mapping $F: R\to R^+$ is called a *distribution function* if it is nondecreasing, ft-continuous with inf $F=0$ and sup $F=1$.

We denote by $\mathcal{L}$ the set of all distribution functions.

Definition 1.2. A *probabilistic metric space* (briefly *PM-space*) is an ordered pair $(S,\mathcal{F})$, where is a set and $\mathcal{F}$ is a function defined on $S\times S$ into $\mathcal{L}$ (we shall denote $\mathcal{F}(p,q)$ by $F_{p,q}$) satisfying

(i) $F_{p,q}(0)=0$,

(ii) $F_{p,q}(x)=1$ if and only if $p=q$,

(iii) $F_{p,q}(x)=F_{q,p}(x)$,

(iv) if $F_{p,q}(x)=1$ and $F_{q,r}(y)=1$, then $F_{p,r}(x+y)=1$.

Definition 1.3. A function $\Delta: [0,1]\times[0,1]\to[0,1]$ is a *Δ-norm* if it satisfies

(I) $\Delta(a,1)=a$, $\Delta(0,0)=0$,

(II) $\Delta(a,b)=\Delta(b,a)$,

(III) $\Delta(c,d)\geq\Delta(a,b)$ for $c\geq a$, $d\geq b$,

(IV) $\Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$.

Definition 1.4. A *Menger space* is a triple $(S,\mathcal{F},\Delta)$, where $(S,\mathcal{F})$ is a PM-space and *Δ-norm* satisfies the following triangle inequality:

(iv′) $F_{p,r}(x+y)\geq\Delta(F_{p,q}(x),F_{q,r}(y))$

or all p,q,r in S and for all $x\geq 0$, $y\geq 0$.

Definition 1.5. A sequence of points $\{p_n\}$ in a PM-space *converges to* p if for every $\varepsilon>0$ and

$\lambda>0$ there exists an integer $M_{\varepsilon,\lambda}$ such that $F_{p,p_n}(\varepsilon)>1-\lambda$ for all $n\geq M_{\varepsilon,\lambda}$.

Theorem 1.6 ([9]). *If $(X,\mathcal{F},\Delta)$ is a Menger space and Δ is continuous, then the statistical distance function, $\mathcal{F}$, is a lower semicontinuous function of points, i.e., for every fixed x, if $q_n\to q$ and $p_n\to p$, then $\liminf_n F_{p_n,q_n}(x)=F_{p,q}(x)$.*

On the other hand, Sehgal-Bharucha-Reid ([10]) obtained the properties about connection between metric spaces and probabilistic metric spaces.

Theorem 1.7. *If (X,d) is a metric space, then the metric d induces a mapping $\mathcal{F}:X\times X\to\mathcal{L}$, where $\mathcal{F}(p,q)$ is defined by $\mathcal{F}(p,q)x=H(x-d(p,q))$, $x\in R$, where $H(x)=0$ if $x\leq 0$ and $H(x)=1$ if $x>0$. Further, if $\Delta:[0,1]\times[0,1]\to[0,1]$ is defined by $\Delta(a,b)=\min\{a,b\}$, then $(X,\mathcal{F},\Delta)$ is a Menger space. It is complete if the metric d is complete.*

2. The Main Theorems

Now, we are ready to give our main theorems.

Theorem 2.1. *Let $(X,\mathcal{F},\Delta)$ be a complete Menger space with continuous Δ-norm: $\Delta(x,x)\geq x$ for each $x\in[0,1]$ and P,Q,S and T be mappings from X into itself satisfying the following conditions:*

(1) $F_{Px,Qy}(t)\geq\min\{F_{x,Px}(t/k),\ F_{y,Qy}(t/k),\ F_{x,Qy}(t/k),\ F_{y,Sx}(t/k),\ F_{x,Ty}(t/k),\ F_{y,Px}(t/k),$ $F_{Px,Sx}(t/k),\ F_{Qy,Ty}(t/k),\ F_{Px,Ty}(t/k),\ F_{Sx,Qy}(t/k),\ F_{Sx,Ty}(t/k)\}$ *for all x,y in X,*

(2) $ST=TS$, $PS=SP$, $PT=TP$, $QS=SQ$ *and* $QT=TQ$,

(3) *there exists a sequence $\{x_n\}_{n\in N}$ in X such that $PTx_{2n}=TSx_{2n+1}$, $QSx_{2n+1}=TSx_{2n+2}$, $Tx_{2n}=TSx_{2n-1}$, $Sx_{2n-1}=TSx_{2n-2}$ for all $n\in N$,*

(4) *S and T are continuous.*

Then P,Q,S and T have a unique common fixed point in X.

Proof. Let $TS=ST=A$. From (1), we have

$$F_{Ax_{2n+1},Ax_{2n}}(t)=F_{PTx_{2n},QSx_{2n-1}}(t)\geq\min\{F_{Tx_{2n},PTx_{2n}}(t/k),\ F_{Sx_{2n-1},QSx_{2n-1}}(t/k),\ F_{Tx_{2n},QSx_{2n-1}}(t/k),$$
$$F_{Sx_{2n-1},STx_{2n}}(t/k),\ F_{Tx_{2n},TSx_{2n-1}}(t/k),\ F_{Sx_{2n-1},PTx_{2n}}(t/k),\ F_{PTx_{2n},STx_{2n}}(t/k),\ F_{QSx_{2n-1},TSx_{2n-1}}(t/k),$$
$$F_{PTx_{2n},TSx_{2n-1}}(t/k),\ F_{STx_{2n},QTx_{2n}}(t/k),\ F_{STx_{2n},TSx_{2n-1}}(t/k)\}=\min\{F_{Ax_{2n-1},Ax_{2n+1}}(t/k),\ F_{Ax_{2n-2},Ax_{2n}}(t/k),$$
$$F_{Ax_{2n-1},Ax_{2n}}(t/k),\ F_{Ax_{2n-2},Ax_{2n}}(t/k),\ F_{Ax_{2n-1},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-2},Ax_{2n+1}}(t/k),\ F_{Ax_{2n+1},Ax_{2n}}(t/k),$$
$$F_{Ax_{2n},Ax_{2n-1}}(t/k),\ F_{Ax_{2n+1},Ax_{2n-1}}(t/k),\ F_{Ax_{2n},Ax_{2n+1}}(t/k),\ F_{Ax_{2n},Ax_{2n-1}}(t/k)\},$$

which on simplication gives

$$F_{Ax_{2n+1},Ax_{2n}}(t)\geq F_{Ax_{2n},Ax_{2n-1}}(t/k).$$

Similarly, $F_{Ax_{2n},Ax_{2n-1}}(t)=F_{Ax_{2n-1},Ax_{2n}}(t)=F_{PTx_{2n-2},QSx_{2n-1}}(t)\geq\min\{F_{Ax_{2n-3},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-2},Ax_{2n}}(t/k),$ $F_{Ax_{2n-3},Ax_{2n}}(t/k),\ F_{Ax_{2n-2},Ax_{2n-2}}(t/k),\ F_{Ax_{2n-1},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-2},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-1},Ax_{2n-2}}(t/k),$ $F_{Ax_{2n},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-1},Ax_{2n-1}}(t/k),\ F_{Ax_{2n-2},Ax_{2n}}(t/k),\ F_{Ax_{2n-2},Ax_{2n-1}}(t/k)\}=F_{Ax_{2n-1},Ax_{2n-2}}(t/k)$.

Hence, in general, $F_{Ax_{n+1},Ax_n}(t)$, $F_{Ax_n,Ax_{n-1}}(t/k),\cdots,$ $F_{Ax_1,Ax_0}(t/k^n)$ for all $n\in N$.

Thus $\{Ax_n\}$ is a Cauchy sequence. Since X is complete, $\{Ax_n\}$ converges to some point z in X, and $\{PTx_{2n}\}$ and $\{QSx_{2n+1}\}$ being the subsequences of $\{Ax_n\}$ converge to the same point z. Therefore, by the conditions (2) and (4),

$$PTSx_{2n_i}=SPTx_{2n_i}\to Sz,\quad STSx_{2n_i}\to Sz,$$
$$QTSx_{2n_i+1}=TQSx_{2n_i+1}\to Tz,\quad TTSx_{2n_i+1}\to Tz,$$

$$TSx_{2n_i}=STx_{2n_i}=STSx_{2n_i}\to Sz \text{ and } TSx_{2n_j+1}=TTSx_{2n_j}\to Tz$$

for some subsequences $\{n_i\}_{i\in N}$ and $\{n_j\}_{j\in N}$ of $\{n\}_{n\in N}$. Hence it follows from $F_{PTSx_{2n_i},QTSx_{2n_j+1}}(t)\geq \min\{F_{TSx_{2n_i},PTSx_{2n_i}}(t/k),\ F_{TSx_{2n_j+1},QTSx_{2n_j+1}}(t/k),\ F_{TSx_{2n_i},QTS_{2n_j+1}}(t/k),\ F_{TSx_{2n_j+1},STSx_{2n_i}}(t/k),$ $F_{TSx_{2n_i},TTSx_{2n_j+1}}(t/k),\ F_{TSx_{2n_j+1},PTSx_{2n_i}}(t/k),\ F_{PTSx_{2n_i},STSx_{2n_i}}(t/k),\ F_{QTSx_{2n_j+1},TTSx_{2n_j+1}}(t/k),$ $F_{PTSx_{2n_i},TTSx_{2n_j+1}}(t/k),\ F_{STSx_{2n_i},QTSx_{2n_j+1}}(t/k)\ F_{STSx_{2n_i},TTSx_{2n_j+1}}(t/k)\}$ and Theorem 1.6 that

$$F_{Sz,Tz}(t)\geq\min\{1,1,F_{Sz,Tz}(t/k),\ F_{Tz,Sz}(t/k),\ F_{Sz,Tz}(t/k),\ F_{Tz,Sz}(t/k),1,1,F_{Sz,Tz}(t/k),\ F_{Sz,Tz}(t/k),\ F_{Sz,Tz}(t/k)\}$$

which means that $Sz=Tz$.

By the same method, we have $Sz=Tz=Pz=Qz$. Furthermore, from

$$F_{PTx_{2n},Qz}(t)\geq min\{F_{Tx_{2n},PTx_{2n}}(t/k),\ F_{z,Qz}(t/k),\ F_{Tx_{2n},Qz}(t/k),$$
$$F_{z,STx_{2n}}(t/k),\ F_{Tx_{2n},Tz}(t/k),\ F_{z,PTx_{2n}}\ (t/k),\ F_{PTx_{2n},STx_{2n}}(t/k),$$
$$F_{Qz,Tz}(t/k),\ F_{PTx_{2n},Tz}(t/k),\ F_{STx_{2n},Qz}(t/k),\ F_{STx_{2n},Tz}(t/k)\}$$

it follows that $F_{z,Qz}(t)\geq min\{1,\ F_{z,Qz}(t/k),\ F_{z,Qz}(t/k),\ F_{z,Qz}(t/k),\ F_{z,Qz}(t/k),\ 1,\ F_{z,Qz}(t/k),\ 1,$ $F_{z,Qz}(t/k),\ 1,1\}$, which means $z=Qz$. Hence $z=Qz=Pz=Tz=Sz$. The uniqueness of the common fixed point z of mappings P,Q,S and T follows easily.

In Theorem 2.1, if P and Q are commuting in stead of the commutativity of S and T, we also obtain the following theorem:

Theorem 2.2. *Let $(X,\mathcal{F},\Delta)$ be a complete Menger space with continuous Δ-norm: $\Delta(x,x)\geq x$ for each $x\in[0,1]$ and P,Q,S and T be mappings from X into itself satisfying the condition* (1) *and the following conditions*:

(5) $PQ=QP,\ PS=SP,\ PT=TP,\ QS=SQ$ *and* $QT=TQ$,

(6) *there exists a sequence $\{x_n\}_{n\in N}$ in X such that* $SQx_{2n+1}=QPx_{2n},\quad TPx_{2n+2}=QPx_{2n+1},\quad Qx_{2n+1}=QPx_{2n}, Px_{2n}=QPx_{2n-1}$ *for all* $n\in N$.

(7) *S and T are continuous.*

Then P,Q,S and T have a unique common fixed point in X.

As immediate consequence of Theorem 2.1, we obtain the following:

Corollary 2.3 ([3]). *Let $(X,\mathcal{F},\Delta)$ be a complete Menger space with continuous Δ-norm: $\Delta(x,x)\geq x$ for each $x\in[0,1]$, and S and T be continuous mappings of X into X. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $SX\cap TX$ which commutes with S and T satisfies the following two conditions*:

(8) $F_{Ax,Ay}\ (t)\geq F_{Sx,Ty}(t/k)$, *for every* $t>0$, *where* $k\in(0,1)$,

(9) *there exists a sequence $\{x_n\}_{n\in N}$ in X such that*

$$Ax_{2n-2}=Sx_{2n-1},\ Ax_{2n-1}=Tx_{2n} \text{ for every } n\in N.$$

Indeed, S, T and A then have a unique common fixed point in X.

Theorem 2.4. *Let $(X,\mathcal{F},\Delta)$ be a complete Menger space with continuous Δ-norm: $\Delta(x,x)\geq x$ for each $x\in[0,1]$, and P,Q,S and T be mappings from X into itself satisfying the conditions* (2), (4) *and the following conditions*:

(10) *there exist positive integers p,q,s,t and a number $k\in(0,1)$ such that* $F_{P^px,Q^qy}\ (t)\geq$

$\min\{F_{x,P^px}(t/k),\ F_{y,Q^qy}(t/k),\ F_{x,Q^qy}(t/k),\ F_{y,S^sx}(t/k),\ F_{x,T^ty}(t/k),\ F_{y,P^px}(t/k),\ F_{P^px,S^sx}(t/k),$
$F_{Q^qy,T^ty}(t/k),\ F_{P^px,T^ty}(t/k),\ F_{S^sx,Q^qy}(t/k),\ F_{S^sx,T^ty}(t/k)\}$

for all x, y *in* X,

(11) *there exists a sequence* $\{x_n\}_{n\in N}$ *in* X *such that* $P^pT^tx_{2n}=T^tS^sx_{2n+1}$. $Q^qS^sx_{2n+1}=T^tS^sx_{2n+2}$, $S^sx_{2n-1}=T^tS^sx_{2n-2}$, $T^tx_{2n}=T^tS^sx_{2n-1}$ *for all* $n\in N$.

Then P, Q, S *and* T *have a unique common fixed point in* X.

Proof. Since P commutes with each of S and T, P^p also commute with each of S^s and T^t. Similarly Q^q commutes with each of S^s and T^t, and S^s commutes with T^t. Thus Theorem 2.1 pertains to P^p, Q^q, S^s and T^t, so there exists a unique point u in X such that $u=P^pu=Q^qu=S^su=T^tu$. From this $Tu=P^p(Tu)=Q^q(Tu)=S^s(Tu)=T^t(Tu)$. Therefore Tu is a common fixed point of P^p, Q^q, S^s and T^t. Similarly, Su is a common fixed point of P^p, Q^q, S^s and T^t, also $Pu=P^p(Pu)=S^s(Pu)=T^t(Pu)$, so Pu is a common fixed point of P^p, S^s and T^t. Similarly, Qu is a common fixed point of Q^q, S^s and T^t. Thus from (10), we obtain

$$F_{Pu,Qu}(t)\geq F_{Pu,Qu}(t/k).$$

So $Pu=Qu$, Su and Tu are the common fixed points of P^p, Q^q, S^s and T^t. Hence by the uniqueness of u,

$$u=Pu=Qu=Su=Tu.$$

Now, we give common fixed point theorems for families of mappings.

Theorem 2.5. *Let* $(X,\mathcal{F},\Delta)$ *be a complete Menger space with continuous* Δ*-norm*: $\Delta(x,x)\geq x$ *for each* $x\in[0,1]$. *If* $\{P_i\}$, $\{Q_i\}$, $\{S_i\}$ *and* $\{T_i\}$ $(i=1,2,\cdots,n)$ *are the families of mappings of* S *into itself such that the compositions* $P_1P_2\cdots P_n$, $Q_1Q_2\cdots Q_n$, $S_1S_2\cdots S_n$ *and* $T_1T_2\cdots T_n$ *are satisfying the conditions* (1), (2), (3) *and* (4), *and* $P_iP_j=P_jP_i$, $Q_iQ_j=Q_jQ_i$, $S_iS_j=S_jS_i$ *and* $T_iT_j=T_jT_i$, *then* $\{P_i\}$, $\{Q_i\}$, $\{S_i\}$ *and* $\{T_i\}$ *have a unique common fixed point* z *in* X.

Proof. Let $P=P_1P_2\cdots P_n$, $Q=Q_1Q_2\cdots Q_n$, $S=S_1S_2\cdots S_n$ and $T=T_1T_2\cdots T_n$. Then the theorem can be proved in a way similar to the Theorem 2.2 of [8].

Remark. We will obtain a common fixed point theorem for the families of mappings as the Theorem 2.5 of [8].

Now, we give an extension of Theorem 2.1 to a non-complete Menger space.

Theorem 2.6. *Let* $(X,\mathcal{F},\Delta)$ *be a Menger space with continuous* Δ*-norm*: $\Delta(x,x)\geq x$ *for each* $x\in[0,1]$, *and* P, Q, S *and* T *be mappings from* X *into itself satisfying the conditions* (1), (2), (3) *and the following conditions*:

(12) *the sequence* $\{TSx_n\}$ *has subsequences converging to a point* z *in* X,

(13) S *and* T *are continuous at* z.

Then z *is the unique common fixed point of* P, Q, S *and* T.

Proof. The proof of Theorem 2.6 is contained in the process of the proof of Theorem 2.1.

As an application of Theorem 2.1, we establish the following theorem:

Theorem 2.7. *Let* $(X,\mathcal{F},\Delta)$ *be a complete Menger space with continuous* Δ*-norm*: $\Delta(x,x)\geq x$ *for* *each* $x \in [0, 1]$, *and* P, Q, S *and* T *be mappings from the product space* $X \times X$ *into* X *satisfying the conditions* (2) *and* (4), *and the following conditions*:

(*a*) $F_{P(x,y),Q(x',y')}(t) \geq \min \{F_{(x,y),P(x,y)}(t/k),\ F_{(x',y'),Q(x',y')}(t/k),\ F_{(x,y),Q(x',y')}(t/k),\ F_{(x',y'),S(x,y)}(t/k),\ F_{(x,y),T(x',y')}(t/k),\ F_{(x',y'),P(x,y)}(t/k),\ F_{P(x,y),S(x,y)}(t/k),\ F_{Q(x',y'),T(x',y')}(t/k),\ F_{P(x,y),T(x',y')}(t/k),\ F_{S(x,y),Q(x',y')}(t/k),\ F_{S(x,y),T(x',y')}(t/k)\}$ *for all* x, y, x', y' *in* X *and* $k \in (0, 1)$.

(*b*) *there exists a sequence* $\{x_n\}_{n \in N}$ *in* X *such that* $P(T(x_{2n}, y), y) = T(S(x_{2n+1}, y), y)$, $QS(x_{2n+1}, y), y) = T(S(x_{2n+2}, y), y)$, $S(x_{2n-1}, y) = T(S(x_{2n-2}, y), y)$, $T(x_{2n}, y) = T(S(x_{2n-1}, y), y)$ *for all* $n \in N$. *Then there exists a unique point* a *in* X *such that* $a = P(a, y) = Q(a, y) = S(a, y) = T(a, y)$ *for all* y *in* X.

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