韓國數學敎育學會誌<數學敎育>
1985. 6, Vol. XXIII, No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1985, Vol. XXIII, No. 2

Generic Submanifolds Satisfying the Cartan Condition of an Odd-Dimensional Sphere

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0. Introduction

A submanifold M of a Sasakian manifold M^{2m+1} is called a *generic* (an *antiholomorphic*) if the normal space $N_p(M)$ of M at any point $p \in M$ is mapped into the tangent space $T_p(M)$ by action of the structure tensor F of the ambient manifold M^{2m+1}, that is, $FN_p(M) \subset T_p(M)$ for each point $p \in M$.

The main purpose of the present paper is to study complete generic submanifolds of an odd-dimensional sphere $S^{2m+1}(1)$ which hold Cartan condition.

In characterizing the submanifolds, we shall use the following Theorem A.

Theorem A ([7]). *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the mean curvature vector of M is parallel in the normal bundle and if the structure induced on M is normal, then M is*

$$S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N),$$

where $p_1, \cdots, p_N$ are odd numbers $\geqq 1$, $r_1{}^2 + \cdots + r_N{}^2 = 1$, $N = 2m+1-n$, $S^p(r)$ being p-dimensional sphere with radius $r > 0$.

1. Preliminaries

Let $S^{2m+1}(1)$ be a $(2m+1)$-dimensional unit sphere covered by a system of coordinate neighborhoods $\{U : y^h\}$ and $(F_j{}^h, G_{ji}, V^h)$ the set of structure tensor of $S^{2m+1}(1)$, that is, $F_j{}^h$ being the Sasakian structure tensor of type $(1,1)$, G_{ji} the Riemannian metric tensor of $S^{2m+1}(1)$ and V^h the Sasakian structure vector, in the sequal, the indices h, i, j and k run over the range $\{1, 2, 3, \cdots, (2m+1)\}$.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhods $\{V : x^a\}$ and isometrically immersed in $S^{2m+1}(1)$ by the immersion $i : M \to S^{2m+1}(1)$. We identify $i(M)$ with M itself and represent the immersion locally by $y^h = y^h(x^a)$, throughout this paper the indices a, b, c, d and e run over the range $\{1, 2, \cdots, n\}$. If we put $B_a{}^h = \partial_a y^h$, $\partial_a = \partial / \partial x^a$, then $B_a{}^h$ are n linearly independent vectors of $S^{2m+1}(1)$ tangent to M.

Denoting by g_{cb} the fundamental metric tensor of M, we have

$$g_{cb} = B_c{}^j B_b{}^i G_{ji}, \tag{1.1}$$

because the immersion is isometric. We represent by $C_x{}^h$ $(p = 2m+1-n)$ mutually orthogonal unit normals to M. Then $G_{ji} B_b{}^j C_x{}^i = 0$ and $G_{ji} C_x{}^j C_y{}^i = g_{xy}$, g_{xy} being the fundamental metric tensor of the normal bundle. In what follows we denote by p the codimension of M and the indices x, y, z, u, v and w run over the range $\{1^*, 2^*, \cdots, p^*\}$.

We now assume that M is generic submanifold of $S^{2m+1}(1)$, we can put in each coordinate neighborhood

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h - f_b{}^x C_x{}^h, \qquad F_i{}^h C_x{}^i = f_x{}^a B_a{}^h, \tag{1.2}$$

where $f_b{}^a$ is a tensor field of type (1, 1) defined on M, $f_c{}^x$ a local 1-form for each fixed index x and $f_x{}^a = f_c{}^y g^{ca} g_{yx}$. Also we can put the Sasakian structure vector V^h of the form

$$V^h = f^a B_a{}^h + f^x C_x{}^h, \tag{1.3}$$

f^a and f^x being vector fields defined on M and normal bundle of M respectively.

Now applying the operator F to (1.2) and (1.3) and using the definition of the Sasakian structure tensor, we easily verify that ([3], [4], [5], [6], [7])

$$\left.\begin{array}{l} f_c{}^e f_e{}^a = -\delta_c^a + f_c{}^x f_x{}^a + f_c f^a, \quad f_c{}^e f_e{}^x = -f_c f^x, \\ f^e f_e{}^a = -f^x f_x{}^a, \quad f_x{}^e f_e{}^y = \delta_x^y - f_x f^y, \quad f_e f^e + f_x f^x = 1, \\ f^e f_e{}^y = 0, \quad g_{de} f_c{}^d f_b{}^e = g_{cb} - f_c{}^x f_{bx} - f_c f_b, \end{array}\right\} \tag{1.4}$$

where $f_c = f^e g_{ce}$ and $f_x = f^y g_{yx}$.

Denoting by ∇_c the operator of van der Waerden-Borotolotti covarient differentiation with respect to the Christoffel symbols formed with g_{cb}, it well known that ([3], [4], [5])

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{1.5}$$

$$\nabla_c f_b{}^x = g_{cb} f^x + h_{ce}{}^x f_b{}^e, \tag{1.6}$$

$$\nabla_c f_b = f_{cb} + h_{cb}{}^x f_x, \tag{1.7}$$

$$\nabla_c f^x = -f_c{}^x - h_{ce}{}^x f^e, \tag{1.8}$$

$$h_{cex} f^{ey} = h_{ce}{}^y f_x{}^e, \tag{1.9}$$

where $h_{cb}{}^x$ is the second fundamental tensor of M and $h_c{}^a{}_x = h_{cb}{}^y g_{yx} g^{ba}$, $(g^{ba}) = (g_{ba})^{-1}$.

The aggregate $(f_c{}^a, g_{cb}, f_c{}^x, f^a, f^x)$ satisfying (1.4) is said to be *normal* (*partially integrable*) if

$$h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e = 0, \tag{1.10}$$

$$f_c{}^e \nabla_e f_b{}^x - f_b{}^e \nabla_e f_c{}^x - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^x - (\nabla_c f_b - \nabla_b f_c) f^x = 0 \tag{1.11}$$

hold respectively ([3], [4], [6]).

Since $S^{2m+1}(1)$ is unit sphere, the equations of Gauss, Codazzi and Ricci are respectively

$$K_{dcb}{}^a = \delta_d^a g_{cb} - \delta_c^a g_{db} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x, \tag{1.12}$$

$$\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x = 0, \tag{1.13}$$

$$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y, \tag{1.14}$$

$K_{dcb}{}^a$ and $K_{dcy}{}^x$ being curvature tensor of M and the normal connection of M respectively. We have from (1.12)

$$K_{cb} = (n-1) g_{cb} + h_x h_{cb}{}^x - h_{ce}{}^x h_b{}^e{}_x, \tag{1.15}$$

K_{cb} being the Ricci tensor of M.

A submanifold M holds Cartan condition if it satisfies

$$\nabla_d \nabla_c K_{ba} - \nabla_c \nabla_d K_{ba} = 0. \tag{1.16}$$

If $K_{dcy}{}^x = 0$, that is,

$$h_{de}{}^x h_c{}^e{}_y = h_{ce}{}^x h_d{}^e{}_y, \tag{1.17}$$

then the normal connection of M is said to be *flat*.

2. Tangential generic submanifolds of an odd-dimensional sphere

In this section we assume that the generic submanifold M of $S^{2m+1}(1)$ is tangent to the structure vector field V^h, then we have

$$h_{ce}{}^x f^e = -f_c{}^x. \tag{2.1}$$

Transvecting (1.15) with f^c and using (2.1), we have

$$K_{be} f^e = (n-1) f_b + h_{be}{}^x f_x{}^e - h_x f_b{}^x. \tag{2.2}$$

Differentiating (2.2) covariantly along M and substituting (1.6) and (1.7), we get

$$(\nabla_c K_{be}) f^e + K_{be} f_c{}^e = (n-1) f_{cb} + (\nabla_c h_{be}{}^x) f_x{}^e + h_{be}{}^x h_{cax} f^{ea} - (\nabla_c h_x) f_b{}^x + h_x h_{ce}{}^x f_b{}^e. \tag{2.3}$$

Differentiating (2.3) covariantly and taking the skew-symmetric part, we obtain

$$\begin{aligned} K_{db} f_c - K_{cb} f_d - K_{be}(h_d{}^e{}_x f_c{}^x - h_c{}^e{}_x f_d{}^x) \\ = (n-1)(g_{db} f_c - g_{cb} f_d - h_{db}{}^x f_{cx} + h_c{}^e{}_x f_d{}^x) \\ - (g_{cb} h_{de}{}^x f_x{}^e - g_{db} h_{ce}{}^x f_x{}^e + h_d{}^a{}_y h_{cb}{}^y h_{ae}{}^x f_x{}^e - h_c{}^a{}_y h_{db}{}^y h_{ae}{}^x f_x{}^e) \\ + (g_{cb} h_x f_d{}^x - g_{db} h_x f_c{}^x + h_x h_{cb}{}^y h_{de}{}^x f_y{}^e - h_x h_{db}{}^y h_{ce}{}^x f_y{}^e) \end{aligned}$$

because of (1.9), (1.13), (1.16) and (1.17).

Transvecting the last equation with f^c and taking the symmetric part with respect to the indices d and b, we have

$$K_{db} = (n-p-1) g_{db} + (h_x - P_x) h_{db}{}^x,$$

where $P_x = h_{cb}{}^y f_y{}^c f_x{}^b$.

Comparing this with (1.15), we have

$$h_{de}{}^x h_b{}^e{}_x = P_x h_{db}{}^x + p g_{db}. \tag{2.4}$$

Transvecting this with f^b and using (2.1), we get

$$h_{de}{}^x f_x{}^e = P_x f_d{}^x - p f_d. \tag{2.5}$$

First of all, we prove

Lemma 2.1. *Le M be a generic submanifold with flat normal connection of an odd-dimensional unit sphere and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If M holds Cartan condition, then the induced structure on M is normal.*

Proof. Computing the length of square of $\nabla_c f_b{}^x + \nabla_b f_c{}^x$, we have

$$\begin{aligned} \frac{1}{2}\|\nabla_c f_b{}^x + \nabla_b f_c{}^x\|^2 = \|\nabla_c f_b{}^x\|^2 + (\nabla_c f_b{}^x)(\nabla^b f_x{}^c) \\ = h_{cb}{}^x h^{cb}{}_x - h_x P^x - np \end{aligned} \tag{2.6}$$

with the aid of (1.4), (1.6), (2.1) and (2.5).

Substituting (2.4) into (2.6), we have

$$\nabla_c f_b{}^x + \nabla_b f_c{}^x = 0.$$

Thus, $h_{ce}{}^x f_b{}^c + h_{be}{}^x f_c{}^c = 0$ holds because of (1.6). This complete the proof of the lemma.

According to Theorem A and Lemma 2.1, we conclude

Theorem 2.2. *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector field defined on $S^{2m+1}(1)$ be tangent to M. If the mean curvature vector of M is parallel in the normal bundle and*

M holds Cartan condition, then M is

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)$$

where $p_1,\cdots,p_N$ *are odd numbers* $\geqq 1$, $r_1^2+\cdots+r_N^2=1$, $N=1m+1-n$.

3. Generic submanifolds with partially integrable structure of $S^{2m+1}(1)$

In this section we assume that the induced structure on M is partially integrable, then we have from (1.11)

$$(h_{cey}f^{ex})f_b{}^y=(h_{bey}f^{ex})f_c{}^y+f_c{}^xf_b-f_b{}^xf_c. \tag{3.1}$$

Transvecting (3.1) with $f_z{}^b$ and using (1.4), we find

$$h_{cez}f^{ex}-(h_{ccy}f^y)f^{cx}f_z=P_{zy}{}^xf_c{}^y-\delta_z{}^xf_c+f_zf^xf_c, \tag{3.2}$$

where

$$P_{zy}{}^x=h_{bcy}f^{ex}f_z{}^b,$$

from which, transvecting f^z,

$$(1-\rho^2)(h_{cey}f^y)f^{ex}=P_{zy}{}^xf^zf_c{}^y-(1-\rho^2)f^xf_c \tag{3.3}$$

where $\rho^2=f_xf^x$.

Substituting this into (3.2), we find

$$(1-\rho^2)h_{cez}f^{ex}=-(1-\rho^2)\delta_z{}^xf_c+[(1-\rho^2)P_{zy}{}^x+f_zP_{yw}{}^xf^w]f_c{}^y. \tag{3.4}$$

Putting $P_{zyx}=P_{zy}{}^wg_{wx}$, then P_{zyx} is symmetric for any index because of (1.9) and (3.3). If we take the skew-symmetric part with respect to the indices x and z and use (1.9), then we obtain from (3.4)

$$(f_zP_{ywx}f^w-f_x\ P_{ywz}f^w)f_c{}^y=0. \tag{3.5}$$

If the function $1-\rho^2$ does not vanish on M, then (2.4) give

$$h_{ce}{}^xf_y{}^e=R_{yz}{}^xf_c{}^z-\delta_y{}^xf_c, \tag{3.6}$$

$$R_{yzx}=P_{yzx}+1/(1-\rho^2)f_zP_{ywx}f^w.$$

Transvecting (3.5) with $f_u{}^c$ and $f_a{}^c$ respectively and taking account of (1.4), we find

$$(f_zP_{ywx}-f_xP_{ywz})f^w=0,$$

this mean that R_{xyz} is symmetric for any index.

Lemma 3.1 ([3], [6]). *Let M be a generic submanifold with flat normal connection of an odd-dimensional unit sphere* $S^{2m+1}(1)$. *If the induced structure on M is partially integrable and the function* $1-f_xf^x$ *does not vanish almost everywhere, then we have* $f^x=0$ *or* $p=1$.

Proof. Since the normal connection of M is flat, by transvecting (1.17) with $f_z{}^c$ and making use of (3.6), we get

$$(R_{wz}{}^xR_{vy}{}^w-R_{wyz}R_v{}^{xw})f_d{}^v=\delta_z{}^x(h_{dey}f^e)-g_{yz}(h_{de}{}^xf^e). \tag{3.7}$$

Transvecting (3.7) with $f^uf_u{}^d$ and using (1.4) and (2.6), we get

$$(R_{wz}{}^xR_{vy}{}^w-R_{wyz}R_v{}^{xw})f^v=g_{yz}f^x-\delta_z{}^xf_y. \tag{3.8}$$

If we transvect (3.7) with f_x and using (3.6),

$$g_{yz}f^x(h_{dex}f^e+f_{dx})=f_z(h_{dey}f^e+f_{dy}), \tag{3.9}$$

from which, contract with respect to y and z

$$(p-1)(h_{dex}f^c+f_{dx})=0.$$

the last two relationships give

$$\rho^2(h_{dey}f^e+f_{dy})(p-1)=0.$$

'ransvecting this with f^{dy} and using (1.4) and (3.6), we have $\rho^4(p-1)^2=0$. This complete the roof of the lemma.

If $p=1$, that is, the submanifold M is a hypersurface of $S^{2m+1}(1)$, the structure induced on M atisfying (1.4) becomes the so-called (f, g, u, v, λ)-structure ([1], [2]) where we have put $f_c{}^x=u^c$, $^a=v^a$, $f^x=f_x=\lambda$ and the equation (3.6) reduces to

$$h_{bc}u^c=\alpha u_b-v_b \tag{3.10}$$

rhere $\alpha=R_{yz}{}^x$ and $h_{bc}=h_{be}{}^x$.

Lemma 3.2 ([3]). *Let M be a partially integrable hypersurface of* $S^{2m+1}(1)$ $(m>1)$, *then we ave*

$$\lambda \text{ is constant on } M,$$

$$h_{ce}v^e=-u_c, \tag{3.11}$$

$$(1-\lambda^2)\nabla_c\alpha=\beta u_c-\lambda(\alpha^2+4)v_c, \tag{3.12}$$

$$2\lambda(\alpha^2+4)f_{cb}-\beta(h_{ce}f_b{}^e-h_{be}f_c{}^e)=\lambda/1-\lambda^2[\alpha\beta+2\lambda(\alpha^2+4)](v_bu_c-v_cu_b), \tag{3.13}$$

$$\beta(h-\alpha)+2(m-1)(\lambda^2+4)=0 \tag{3.14}$$

vhere $\beta=u^eV_e\alpha$.

Lemma 3.3. *Let M be a partially integrable hypersurface of* $S^{2m+1}(1)$ $(m>1)$. *If M hold Cartan ondition, then we have* $\lambda=0$.

Proof. If we transvect (1.15) with v^c and u^c, we have

$$K_{be}v^e=2(m-1)v_b+(\alpha-h)u_b, \tag{3.15}$$

$$K_{be}u^e=(2m-2+\alpha h-\alpha^2)u_b+(\alpha-h)v_b. \tag{3.16}$$

Differentiating (3.15) covariantly, we find

$$(\nabla_cK_{be})v^e+K_{be}\nabla_cv^e=2(m-1)\nabla_cv_b+\nabla_c(\alpha-h)u_b+(\alpha-h)\nabla_cu_b \tag{3.17}$$

and differentiating (3.17) covariantly and transvecting u^bu^c, we easily verify that

$$\lambda\alpha(\alpha-h)=0. \tag{3.18}$$

This implies $\lambda=0$ or $\alpha(\alpha-h)=0$ because λ is constant. Therefore if $\alpha(\alpha-h)=0$, we see that $\alpha=0$ or $\alpha=h$. In these two case, we see from (3.12) and (3.14) that $\lambda=0$. This complete the proof of the lemma.

According to Theorem 2.2, Lemma 3.1 and Lemma 3.3, we conclude

Theorem 3.4. *Let M be a n-dimensional complete generic submanifold of an odd-dimensional unit sphere* $S^{2m+1}(1)$ *with flat normal connection. Suppose that the induced structure on M is partially integrable, the function* $1-f_xf^x$ *does not vanish almost everywhere and the mean curvature vector of M is parallel in the normal bundle. If M holds Cartan condition, then M is*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)$$

where $p_1,\cdots,p_N$ *are odd-numbers* $\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1, N=2m+1-n$.

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