國數學敎育學會誌<數學敎育>
985. 6, Vol. XXIII, No. 2

Journal of the Korea Society of Mathematical Education
Jun. 1985, Vol. XXIII, No. 2

On a Local Version of a Levi-Civita Connection

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1. Introduction

We will begin by fixing some notations and standard facts that are necessary in inducing the ıain results;

Let M be an n-dimensional Riemannian manifold with metric g and $O(M)$ the bundle of ortho-ıormal frames over M. Every connection in $O(M)$ determines a connection in the bundle $L(M)$ ›f linear frames. A linear connection of M is a metric connection if it is determined by a connec-ion in $O(M)$ (for the detail, see [6]). Among all the possible metric connections, the most mportant is Riemannian connection (which is also called the Levi-Civita connection) which is given ›y the following theorem.

Theorem A. *Every Riemannian manifold admits a unique metric connection with vanishing torsion.*

In this paper, we derive a local version of Theorem A.

2. Preliminaries

In this section, we introduce some basic notions which are necessary for our purpose.

Let $\langle\,,\rangle$ be the scalar product on $\boldsymbol{R}^n$ defined by $\langle v,w\rangle=v_1w_1+\cdots+v_rw_r-v_{r+1}w_{r+1}-\cdots-v_{r+s}w_{r+s}$. ‚et $O(r,s)$ denote the subgroup of $GL(n,\boldsymbol{R})$ preserving $\langle\,,\rangle$, in the sense that $A\in O(r,s)$ if and ›nly if $\langle Av,Aw\rangle=\langle v,w\rangle$. If η is the $n\times n$ diagonal matrix with diagonal entries $(1,\overset{r}{\cdots},1,\ -1,\overset{s}{\cdots},$ $-1)$, then for $v,w\in\boldsymbol{R}^n$, $\langle v,\ w\rangle=v^t\eta w$, and $A\in O(r,s)$ iff $A^t\eta A=\eta$.

If $t\mapsto A(t)$ is a curve in $O(r,s)$, with $A(0)=I$, then $\langle A'(0)v,w\rangle+\langle v,A'(0)w\rangle=0$. Thus the Lie lgebra of $O(r,s)$ is $\{B\in gl(n,\boldsymbol{R})\,|\,\langle Bv,w\rangle+\langle v,Bw\rangle=0\}=\{B\in gl(n,\boldsymbol{R})\,|\,B^t\eta+\eta B=0\}$.

An orthonormal frame at $x\in M$ is a frame $u\in L(M)_x$, $u:\boldsymbol{R}^n\to T_xM$ such that $g(u(v),\ u(w))=$ $\langle v,w\rangle$. The set of all orthonormal frame at x is denote by $O(M)_x$, and we set $O(M)=\bigcup_{x\in M}O(M)_x$ ınd define Π: $O(M)\to M$ by $\Pi(u)=x$ if $u\in O(M)_x$. Note that if $u\in O(M)_x$ and $A\in O(r,s)$, then $A=u\circ A\in O(M)_x$. Then $\Pi:O(M)\to M$ is a principal fibre bundle with group $O(r,s)$.

Let $T^{p,q}(E,F)$ be the space of multilinear functions f: $\hat{E}\times\overset{p}{\cdots}\times\hat{E}\times E\times\overset{q}{\cdots}\times E\to F$, where $p,q\geq 0$, ›here E,F be the vector spaces over $\boldsymbol{R}$ and $\hat{E}$ the dual of E. Let $A^k(E,F)$ be the subspace of $T^{o,k}(E,F)$ consisting of functions f, such that $f(u_1,\cdots,u_k)$ is antisymmetric in $u_1,\cdots,u_k\in E$. Then he canonical 1-form on $O(M)$ is the $\boldsymbol{R}^n$-valued form $\theta\in A'(O(M),\boldsymbol{R}^n)$ defined by $\theta(X_u)=$ $u^{-1}(\Pi_*(X_u))$ for $X_u\in T_uO(M)$. Note that this 1-form is the restriction of a form $\bar{\theta}\in A'(L(M),\ \boldsymbol{R}^n)$ efined by the same equation (for $X_u\in T_uL(M)$). We now define the torsion form Φ of a connec-ion w on $O(M)$ by $\Phi=D\theta$ (where D is the convariant differential).

Let $\bar{\Lambda}^k(P, V)$ be the space of V-valued differential k-forms φ on P such that

(a) For $X_1, \cdots, X_k \in T_pP$, we have
$\varphi(R_{g*}X_1, \cdots\cdots, R_{g*}X_k) = g^{-1} \circ \varphi(X_1, \cdots, X_k)$.

(b) If one of X_i is vertical, then $\varphi(X_1, \cdots, X_k) = 0$.

3. Main Results

We define a local section of a principal bundle $\Pi : P \to M$ with group G to be a map $\sigma : U \to P$ ($U \subset M$, open) such that $\Pi \circ \sigma = 1_U$.

Then we have

Lemma 1. *There is a natural correspondence between local sections and local trivialization.*

Proof. If $\sigma : U \to P$ is a local section, then define $T_u : \Pi^{-1}(U) \to U \times G$ by $T_u(\sigma(x)g) = (x, g)$. Conversely, given a local trivialization $T_u : \Pi^{-1}(U) \to U \times G$, define $\sigma : U \to P$ by $\sigma(x) = T_u^{-1}(x, e)$, where e is the identity of G. Then clearly σ is a local section.

Let $E_1, \cdots, E_n$ be orthonormal vector fields defined on some open $U \subset M$, and let $\bar{\varphi}^1, \cdots, \bar{\varphi}^n$ be the 1-forms dual to $E_1, \cdots, E_n$ (i.e., $\bar{\varphi}^i(E_j) = \delta_{ij}$). Then we have the main results of this paper.

Main Theorem. *There is a unique matrix $\bar{\theta} = (\bar{\theta}_{ij})$ of real-valued 1-forms $\bar{\theta}_{ij} \in \Lambda'(U, \boldsymbol{R})$ such that (a) $\bar{\theta}^t \eta + \eta \bar{\theta} = 0$, where $\eta = \mathrm{diag}(1, \overset{p}{\cdots}, 1, -1, \overset{s}{\cdots}, -1)$.*

(b) $d\bar{\varphi}^i = -\sum_j \bar{\theta}_{ij} \wedge \bar{\varphi}^j$.

Proof. Let $e_1, \cdots, e_n$ be the standard basis of $\boldsymbol{R}^n$, Then we can define a local section $\sigma : U \to O(M)$ by letting $\sigma(x) : \boldsymbol{R}^n \to T_xM$ be given by $\sigma(x)(e_i) = E_{ix}$ for $i = 1, \cdots, n$. If φ is the canonical 1-form then $\sigma^*(\varphi)(E_i) = \varphi(\sigma_* E_i) = \sigma(x)^{-1}(\Pi_* \sigma_* E_i) = \sigma(x)^{-1}(E_i) = e_i = (\bar{\varphi}^1(E_i), \cdots, \bar{\varphi}^n(E_i)) = \bar{\varphi}(E_i)$ for $i = 1, \cdots, n$. Hence $\sigma^*(\varphi) = \bar{\varphi}$.

If θ is the Levi-Civita connection on $O(M)$, then $d\varphi = -\theta \wedge \varphi$ and applying σ^*, we have $d\bar{\varphi} = -\sigma^*(\theta) \wedge \bar{\varphi}$. Thus, setting $\bar{\theta} = \sigma^*(\theta)$, we have (b), and since θ is $O(r, s)$-valued, we have (a).

(Uniqueness) Let $\bar{\theta}'$ satisfy (a) and (b). Then $\bar{\theta}'$ induces a connection θ' on $\Pi^{-1}(U)$ such that $\sigma^*\theta' = \bar{\theta}'$. Now $\sigma^*(\Phi) = \sigma^*(d\varphi + \theta' \wedge \varphi) = d\bar{\varphi} + \bar{\theta}' \wedge \bar{\varphi} = 0$ so $\Phi \in \bar{\Lambda}^2(\Pi^{-1}(U), O(r, s))$ vanishes on $\sigma(U)$, and hence throughout $\Pi^{-1}(U)$. Thus θ' must be the Levi-Civita connection, and $\bar{\theta}' = \sigma^*\theta' = \sigma^*\theta = \bar{\theta}$.

References

1. Bang-Yen Chen and Tadashi Nagano, Totally Geodesic Submanifolds of Symmetric Spaces 1, *Duke Math. Journal*, Vol. 44, No. 4, 1977.
2. Bishop, R.L. and Crittenden, R.J., *Geometry of Manifolds*, Academic Press, 1964.
3. Boothby, W.M., *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, 1975.
4. Heng-Lung Lai and Huei-Shyong Lue, Scalar Curvature of Lie Groups, *Proceedings of A.M.S.*, Vol. 81, No. 2, 1981.
5. Husemoller, D., *Fibre Bundles*, 2nd ed., Springer-Verlag, New York, 1975.
6. Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry*, Vol. I, II, Interscience Publishes, 1969.

Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. I, II, III, Publish or Perish, Inc., Berkeley, 1979.

Milnor, J. and J. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, N.J., 1974.