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On O-Semimetrizability of Topological Spaces

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Many of the generalized metric spaces can be characterized by *separation properties.* The main method in this paper is another chracterization of metrizable spaces by a symmetric countable open covering map g.

1. Preliminaries

For a subset S of a space, we will denote the closure, and the complement of S by S^- and S^c, respectively. Throughout this paper, all spaces will be T_1- spaces.

Let X be a space and g a function from $N\times X$ (N=the set of positive integers) into the the topology of X such that

$$x\in g(n,x) \text{ and } g(n+1,x)\subset g(n,x)$$

for each $(n,x)\in N\times X$. We call such a function a *COC-map* (=*countable open covering map*). For any subset S of X, we denote

$$g(n,S)=\cup\{g(n,x):x\in S\}.$$

Let $\mathscr{A}, \mathscr{B}$ be some families of subsets of X. Consider the following separation properties on a *COC*-map g.

Definition 1.1. For each $A\in\mathscr{A}$, $B\in\mathscr{B}$ disjoint, if there exists an $n\in N$ such that

1. $A\cap g(n,B)=\phi$, then g *separates* $\mathscr{B}$ from $\mathscr{A}$,
2. $A\cap g^2(n,B)=\phi$, then g *separates doubly* $\mathscr{B}$ from $\mathscr{A}$,
3. $A\cap g(n,B)^-=\phi$, then g *separates regulary* $\mathscr{B}$ from $\mathscr{A}$.

Definition 1.2. A space X is *semimetrizable* if there exists a real valued function d on $X\times X$ such that (1) $d(x,y)=d(y,x)\geqslant 0$, (2) $d(x,y)=0$ if and only if $x=y$, (3) for $M\subset X$, $x\in M^-$ if and only if $d(x,M)=inf\{d(x,y): y\in M\}=0$. If in addition, d satisfies (4) for every $\varepsilon>0$ and $x\in X$, $S_d(x;\varepsilon)=\{y\in X: d(x,y)<\varepsilon\}$ is an open subset of of X, then X it said to be *o-semimetrizable.*

Lemma 1.3. *A space X is o-semimetrizable if and only if for each $x\in X$, there is a symmetric COC-map g such that if $x\in g(n,x_n)$,* then *x is a cluster point of* $\{x_n\}$.

Proof. Let X be a o-semimetrizable space. For each n, take $g(n,x)=S(x;\frac{1}{n})$. Then clearly g is symmetric. Let $x\in g(n,x_n)$ and U be a neighborhood of x. The for some k, $g(k,x)=S(x,\frac{1}{k})\subset U$ and for all $n\geqslant k$, $x_n\in g(n,x)\subset U$. Conversely, for any $x,y\in X$ define a o-semimetric d by $d(x,y)$=the smallest integer n such that $y\notin g(n,x)$. Then d is a o-semimetric.

2. Main Results

Sakong gave a question.

Question. Suppose that X has a symmetric *COC*-map separating regularly points from closed sets. Is X metrizable?

Such a space X is not metrizable, but o-semimetrizable.

Counter Example 2.1. We actually use a space first constructed by Borges [2]. Let X be the set of all points (x,y) of the plane such that (1) $y=0$ and x, $\sqrt{2}/n-x$, $\sqrt{2}/n+x$ are irrational for each positive integer n, or (2) x is rational and $y=\sqrt{2}/n$ for some positive integer n. A base for a topology on X consists of all sets $B((x,y),n)=\{(x,y)\}\cup\{(w,z)\in X: |w-x|<1/n$ and $0\leqslant|z-y|<|w-x|\}$ (i.e., $B((x,y),n)$ is a "butterfly region centered at (x,y) with radius $1/n$ and vertex angle $\pi/4$") for any $(x,y)\in X$ and positive integer n. Actually a "small" neighborhood of a point $(x, \sqrt{2}/n)$ X is just an open interval of rational numbers in the horizontal line passing through $(x, \sqrt{2}/n)$ and containing $(x, \sqrt{2}/n)$. Also the only boundary points of a neighborhood $B((x,0),n)$ are $(x+1/n,0)$ and $(x-1/n,0)$, since the hypotenuses of the "wings" of the butterfly $B((x,0),n)$ contain no paints $(w, \sqrt{2}/n)$ X because of (1). Consequently one immediately sees that X is a regular space. By defining $g(n,(x,y))=B((x,y),n)$, we can see that g separates regularly points from closed. And Borges proved that X is not stratifiable hence X is not metrizable.

Theform 2.2. *A regular space X is o-semimetrizable if and only if it has a symmetric COC-map separating regularly points from closed sets.*

Proof. Let g be a symmetric *COC*-map which the condition in Lemma 1.3. Let F be a closed set not containing x. Since X is regular, there is a neighborhood U of x such that $U^-\cap F=\phi$. Then U^c is also a closed set not containing x. Suppose that $g(n,x)\cap U^c\neq\phi$ for every $n\in N$. Then there are $x_n\in U^c$ so that $x_n\in g(n,x)$. By symmetry, $x\in g(n,x_n)$. Thus x is a cluster point of x_n, which is a contradiction. Since for some $n\in N$, $g(n,x)\subset U$, $g(n,x)^-\subset U^-$. Therefore $g(n,x)^-\cap F=\phi$ for some $n\in N$.

For the converse, let g be a symmetric *COC*-map separating regularly points from closed sets. Suppose that $x\in g(n,x_n)$ for every $n\in N$. Then by symmetry, $x_n\in g(n,x)$ for every $n\in N$. Let U be an open neighborhood of x. Since $x\notin U^c$, there exists $k\in N$ such that $g(k,x)^-\cap U^c=\phi$. Then for every $n\geqslant k$, $x_n\in g(n,x)\subset g(k,x)\subset U$. Therefore x is a cluster point $\{x_n\}$.

Let d be a semimetric for X. The following condition is due to Arhangel'skii.

(K) For any disjoint compact K_1 and K_2 in X, $d(K_1,K_2)>0$.

Definition 2.3. A semimetric satisfying (K) is called a K-semimetric. A space is said to be K-semimetrizable if it is semimetrizable via a K-semimetric.

Theorem 2.4. *A regular space X has a COC-map g_1 which separates doubly points from closed and a COC-map g_2 which separates doubly closed from points. Then X is a K-semimetrizable.*

Proof. Let $g(n,x)=g_1(n,x)\cap g_2(n,x)$ for each $x\in X$. Then clearly g is a *COC*-map which separates doubly points from closed and separates doubly closed from points. Define a semimetric d by $d(x,y)=1/\inf\{j\in N: x\notin g(j,y)$ and $y\notin g(j,x)\}$. Since g separates doubly points from closed, d is ell-defined. Let K_1 and K_2 be disjoint compacta. For each x in K_1, there exists $n(x) \in N$ such ıat $x \notin g(n(x), K_2)$. This implies that $\{X - g(n(x), K_2) : x \in K_1\}$ forms an open cover of the ›mpact set K_1. Let $\{X - g(n(x_i), K_2) : x_i \in K_1, 1 \leqslant i \leqslant k\}$ be a finite subcover of K_1, and $n =$ ıax $\{n(x_i) : 1 \leqslant i \leqslant k\}$. Then $K_1 \cap g(n, K_2) = \phi$. Similarly there exists $n' \in N$ such that $K_2 \cap g(n', K_1)$:ϕ. It follows that $d(K_1, K_2) \geqslant 1/m > 0$, where $m = max\{n, n'\}$.

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