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A Note on the Disconjugacy of Third Order Linear Differential Equation

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1. Introduction

The objective of this dissertation is to study the ***disconjugacy of the third order linear differential equation***

(E) $$L(y)=y'''+P(x)y'+Q(x)y=0,$$

where $P(x)$ and $Q(x)$ are continuous non-negative functions on $[a,\infty)$. We establish conditions which will insure that for each $t\in[a,\infty)$, (E) has a non-trivial solution $y(x)$ with at least three zeros on $[t,\infty)$. The zeros of solutions of (E) are isolated points on $[a,\infty)$, that is, the zeros of a solution of (E) do not have a finite limit point.

The notion of ***disconjugacy*** is originally applied to the second order equation

(S) $$(r(x)y')'+q(x)(y)=0,$$

where (S) is disconjugate on an interval I if no solution has more than one zero on I. Thus disconjugacy of the third order equations is a natural extension from equations of second order. Disconjugacy for nth order equations is defined in an analogous manner.

2. Preliminaries

Let $y(x)$ be a solution of (E). Then $y(x)$ is said to be ***oscillatory*** if the set of zeros of $y(x)$ is not bounded above. This is equivalent to saying that $y(x)$ has infinitely many zeros on $[a,\infty)$. The Solution $y(x)$ is ***non-oscillatory*** if it is not oscillatory, that is, if it has at most a finite number of zeros on $[a,\infty)$.

Let $y(x)$ and $z(x)$ be solutions of (E). Let $z(x)$ be a twice differentiable function on $[a,\infty)$ such that $z''(x)+P(x)z(x)$ is differentiable on this interval. Then, multiplying the third order differential operator $L(y)$ by $z(x)$ and using the rule for differentiating the product of two functions, we obtain

$$z(x)L(y)=\{y(x):z(x)\}'-[(z''(x)+P(x)z(x))'-Q(x)z(x)]y(x) \tag{2.1}$$

where $\{y(x):z(x)\}$ is given by

$$\{y(x):z(x)\}=z(x)y''(x)-z'(x)y'(x)+[z''(x)+P(x)z(x)]y(x). \tag{2.2}$$

Definition 2.1. The third order differential equation

(E*) $$L^*(z)=(z''+P(x)z)'-Q(x)z=0$$

is said to be the ***adjoint*** of (E) and the operator $L^*(z)$ is called the adjoint operator of $L(y)$.

Definition 2.2. Let c be any point on $[a,\infty)$ and let $u_i(x,c)$, $i=1,2$ be the pair of solutions determined by the initial conidtions

$$\text{(a) } u_1(x,c)\ ;\ y(c)=0,\ y'(c)=1,\ y''(c)=0, \tag{2.3}$$
$$\text{(b) } u_2(x,c)\ ;\ y(c)=0,\ y'(c)=0,\ y''(c)=1.$$

The solutions $u_2(x,c)$ and $u_1(x,c)$ are called the first and second principal solutions, respectively, at $x=c$.

Definition 2.3. Let $D_2(y)$ denote the second order differential operator $D_2(y)=y''+P(x)y$. The first and second principal solutions $u_2{}^*(x,c)$ and $u_1{}^*(x,c)$ of (E*) at the point $x=c$, $c\in[a,\infty)$, are determined by the initial conditions

$$\text{(a) } u_1{}^*(x,c)\ ;\ z(c)=0,\ z'(0)=1,\ D_2z(c)=0, \tag{2.4}$$
$$\text{(b) } u_2{}^*(x,c)\ ;\ z(c)=0,\ z'(c)=0,\ D_2z(c)=1.$$

Definition 2.4. A third order linear differential equation is said to be *disconjugate* on an interval I if no solution of the equation has more than two zeros, counting multiplicities, on I.

3. Disconjugacy

Lemma 3.1. *Let (E) be disconjugate on $[a,\infty)$ and let its coefficients satisfy*

(C) $\qquad P(x)\geq 0,\ Q(x)>0$ *and* $P(x)/Q(x)$ *is nondecreasing on* $[a,\infty)$.

If $u_2''(x,a)$ has a zero on $[a,\infty)$ with $x=t_1$ being the first zero of $u_2''(x,a)$ then

(a) *$u_2''(x,a)$ has a second zero $t_2\in(t_1,\infty)$,*

(b) *$u_2'(x,a)$ has exactly one zero $s_1\in(t_1,t_2)$ and $u_2'(x,a)<0$ on (s_1,∞).*

Proof. See J.H. Barrett [4, p.215].

Lemma 3.2. *Let* (E) *be disconjugate on $[a,\infty)$ and let its coefficients satisfy* (C). *Then* $P(x)D_2u_2{}^*(x,a)+Q(x)u_2{}^{*\prime}(x,a)>0$ *on* $[a,\infty)$.

Proof. Since $u_2{}^*(x,a)$ is a solution of the adjoint (E*) of (E), we have

$$[u_2{}^{*\prime\prime}(x,a)+P(x)u_2{}^*(x,a)]'=Q(x)u_2{}^*(x,a).$$

Integrating from a to x, yields

$$u_2{}^{*\prime\prime}(x,a)+P(x)u_2{}^*(x,a)=1+\int_a^x Q(t)u_2{}^*(t,a)\,dt.$$

Therefore

$$u_2{}^{*\prime}(x,a)=(x-a)+\int_a^x\int_a^t Q(s)u_2{}^*(s,a)\,ds\,dt-\int_a^x P(t)u_2{}^*(t,a)\,dt$$
$$=(x-a)+\int_a^x (x-t)Q(t)u_2{}^*(t,a)\,dt-\int_a^x P(t)u_2{}^*(t,a)\,dt.$$

Now,
$$\begin{aligned}P(x)D_2u_2{}^*(x,a)+Q(x)u_2{}^{*\prime}(x,a)&=P(x)+P(x)\int_a^x Q(t)u_2{}^*(t,a)\,dt+Q(x)(x-a)\\&\quad+Q(x)\int_a^x (x-t)Q(t)u_2{}^*(t,a)\,dt\\&\quad-Q(x)\int_a^x P(t)u_2{}^*(t,a)\,dt\\&=P(x)+Q(x)(x-a)+Q(x)\int_a^x (x-t)Q(t)u_2{}^*(t,a)\,dt\\&\quad+\int_a^x [P(x)Q(t)-Q(x)P(t)]u_2{}^*(t,a)\,dt.\end{aligned}$$

Since $P(x)/Q(x)$ is nondecreasing and $u_2^*(x,a)>0$ it follows that

$$P(x)D_2u_2^*(x,a)+Q(x)u_2^{*\prime}(x,a)>0 \text{ on } [a,\infty).$$

Lemma 3.3. *Let* (E) *be disconjugate on* $[a,\infty)$ *and let its coeffients satisfy* (C). *Assume* $u_2''(x,a)$ *has a zero at* t_1. *Then* $u_2''(x,a)$ *has a second zero, say at* t_2, *and* $u_2''(x,a)>0$ *on* (t_2,∞), $a<t_1<t_2$.

Proof. See J.H. Barrett [4, p. 217].

Lemma 3.4. *Let* $(P(x)y')'+Q(x)y=0$ *be disconjugate on* $[a,\infty)$. *If* $\int_a^\infty \frac{1}{P(x)}dx=\infty$, $Q(x)\geq 0$ *with* $Q(x)\not\equiv 0$ *for large* x, *and* $y(x)$ *is any non-trivial solution of* $(P(x)y')'+Q(x)y=0$ *with* $y(a)=0$, *then* $y(x)y'(x)>0$ *on* $[a,\infty)$.

Proof. See McKelvey [2, p. 5].

Theorem 3.5. *If equation* (E) *is disconjugate on* $[a,\infty)$ *and its coefficients satisfy* (C), *then* $u_2''(x,a)>0$ *on* $[a,\infty)$.

Proof. Suppose $u_2''(x,a)$ has a zero at $x=t_1$ on $[a,\infty)$. By Lemma 3.1, $u_2'(x,a)$ has a zero at $x=s_1$ and $u_2'(x,a)<0$ on (s_1,∞). From (2.2) we have $\{u_2^*(x,a) : y(x)\}=k$, constant, for each solution $y(x)$ of (E). In particular, for the solution $u_2(x,a)$ and $u_1(x,a)$, we have

$$\{u_2^*(x,a) : u_2(x,a)\}=\{u_2^*(x,a) : u_1(x,a)\}=0.$$

Thus $u_1(x,a)$ and $u_2(x,a)$ are linearly independent solutions of the second order equation

$$u_2^*(x,a)y''-u_2^*(x,a)y'+(D_2u_2^*(x,a))y=0.$$

Also, since $u_1(x,a)$ and $u_2(x,a)$ are each solutions of (E), we find, upon eliminating the y-term, that $u_1(x,a)$ and $u_2(x,a)$ are solution of

$$(D_2u_2^*(x,a))y'''+Q(x)u_2^*(x,a)y''+[P(x)D_2u_2^*(x,a)+Q(x)u_2^{*\prime}(x,a)]y'=0.$$

Now, from (E*), $D_2u_2^*(x,a)=1+\int^x_a Q(s)u_2^*(s,a)>0$.

Therefore, after dividing by $[D_2u_2^*(x,a)]^2$, we obtain

$$\left[\left(\frac{1}{D_2u_2^*(x,a)}\right)y''\right]'+\frac{[P(x)D_2u_2^*(x,a)+Q(x)u_2^{*\prime}(x,a)]}{(D_2u_2^*(x,a))^2}y'=0.$$

Letting $w=y'$, we have the second order equation

$$\text{(D)}\qquad \left(\frac{1}{D_2u_2^*(x,a)}w'\right)'+\frac{P(x)D_2u_2^*(x,a)+Q(x)u_2^{*\prime}(x,a)}{(D_2u_2^*(x,a))^2}w=0.$$

Clearly, $u_1'(x,a)$ and $u_2'(x,a)$ are solutions of (D) and since $u_1'(t_1,a)=0$, $u_2'(x,a)<0$ on (t_1,∞), (D) is disconjugate on (t_1,∞).

As observed above $D_2u_2^*(x,a)=1+\int_a^x Q(t)u_2^*(t,a)dt\geq 1$, so $\int_a^\infty D_2u_2^*(t,a)dt=\infty$.

By Lemma 3.2, $P(x)D_2u_2^*(x,a)+Q(x)u_2^{*\prime}(x,a)>0$. Quoting Lemma 3.4, we have $u_2'(x,a)u_2''(x,a)>0$ on (t_1,∞) or $u_2''(x,a)<0$, contradicting Lemma 3.3. This completes the proof of Theorem.

References

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