LOCALIZATIONS AND GENERALIZED EVALUATION
SUBGROUPS OF HOMOTOPY GROUPS

JAE-RYONG KIM

D.H. Gottlieb [1] has defined and studied the evaluation subgroups $G_n(X)$ of homotopy groups $\pi_n(X)$ of a topological space $X$. Moreover G.E. Lang, Jr. [8] has proved that $G_n(X_p) \cong G_n(X)_p$ for $p=0$ or a prime, where $X$ is a simple connected finite $CW$-complex, $X_p$ is its localization at $p$ for $p$ prime or 0 and $G_n(X)_p$ is the localization of the group $G_n(X)$.

On the other hand, Moo Ha Woo and the present author [10] have defined subgroups $G_n f(X, A)$ of $\pi_n(X)$ which contain the evaluation subgroups $G_n(X)$ and have generalized the properties of the evaluation subgroups $G_n(X)$. These subgroups $G_n f(X, A)$ will be called the generalized evaluation subgroups.

The purpose of the present paper is to localize the generalized evaluation subgroups $G_n f(X, A)$ and to generalize the results of G.E. Lang, Jr. [8]. Moreover we study the relationships between the group $G_n f(X, A)$ and the genus $G(f)$ of the map $f$.

1. Localizations

Let $C$ be a category and let $P$ be a family of (rational) prime numbers. Let us suppose that we have a full subcategory $C_P$ of $C$ and that we agree to call the object of $C_P$ the $P$-local object of $C$. We will say we have a $P$-localization theory in $C$ if we may associate with each object $X$ of $C$ an object $X_P$ of $C_P$ and a morphism $e=e_P : X \rightarrow X_P$ of $C$ such that, given any morphism $f : X \rightarrow Y$ in $C$ with $Y$ in $C_P$, there exists a unique morphism $\tilde{f} : X_P \rightarrow Y$ in $C_P$ with $\tilde{f} e = f$.

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This is equivalent to asking that the embedding functor $C_p \leq C$ should have a left-adjoint left-inverse $L : C \to C_p$ (so that $LX = X_p$). If we have $P$-localization theory for all prime $P$, we say that we have a localization theory in $C$. We write $g_p$ for $Lg$ for any morphism $g$ in $C$; we call $e$ (or any morphism equivalent to $e$) the $P$-localizing morphism.

General references for the localization theory are [3], [4] and [5]. We review some of these results here.

Assume that spaces $X$ and $A$ are pointed, simple (abelian fundamental groups acting trivially on the homotopy and homology groups), connected finite $CW$-complexes. For a prime $p$ let $Q_p$ be the localization of the integers at the prime $p$ (i.e., $Q_p$ is the subring of the rationals $Q=Q_0$, consisting of those rationals expressible as $a/b$, where $(b, p) = 1$).

We define an abelian group $B$ to be $p$-local if it admits unique division by all primes $q$ relatively prime to $p$.

Now let $A$ be any abelian group. We define its $p$-localization to the abelian group $A_p = A \otimes Q_p$, and the $p$-localizing map is the canonical homomorphism $e : A \to A_p$, given by $e(a) = a \otimes 1$, $a \in A$.

Our definition of a $p$-local nilpotent group $H$ differs from that given above only insofar as we use multiplicative notation in the nilpotent group category and additive notation in the abelian group category. It is then true, but far from immediate, that the category of nilpotent groups admits a localization theory [4]. Moreover, this theory does generalize the theory as above in the sense that, if the abelian group $A$ is regarded as a nilpotent group then its $p$-localization, as a nilpotent group, coincides with its $p$-localization as an abelian group.

We recall that a (pointed) space $X$ is said to be nilpotent if it is of the pointed homotopy type of a $CW$-complex, and if moreover $\pi_1(X)$ is nilpotent and operates nilpotently on the higher homotopy groups of $X$. If $X$ is nilpotent and $A$ is a compact polyhedron, then every component of the function space $X^A$ is nilpotent [2].

We say that a nilpotent space $X$ is $p$-local if all its homotopy groups are $p$-local. A map $e_p : X \to Y$ $p$-localizes $X$ if $Y$ is $p$-local and $e_p^* : [Y, Z] \cong [X, Z]$ for all $p$-local $Z$. It is equivalent that $e_p^* : \pi_n(X) \to \pi_n(Y)$ $P$-localizes $\pi_n(X)$ $n \geq 1$ [4]. That is, $e_p^*$ is a $p$-isomorphism ($p$-injective and $p$-surjective) and $\pi_n(X_p)$ is $p$-local.
We say that a homomorphism \( \phi : A \to B \) from a group \( A \) into a group \( B \) is \( p \)-injective if the kernel of \( \phi \) consists of elements of finite orders prime to \( p \); and that is \( p \)-surjective if, given any \( b \in B \), there exists a positive integer \( n \) prime to \( p \) such that \( b^n \in \text{Im} \phi \).

In particular, every simple, pointed \( CW \)-space \( X \) admits a \( p \)-localization map \( e_p : X \to X_p \) with \( X_p \) a pointed simple \( CW \)-space.

Let \( \hat{\epsilon}_p : (X^A, f) \to (X^A_p, e_p f) \) be defined by \( \hat{\epsilon}_p(g) = e_p g \).

**Theorem 1.1.** If \( A \) is a connected finite \( CW \)-complex and \( X \) is a connected simple \( CW \)-complex, then \( \hat{\epsilon}_p \) \( p \)-localizes [2][8].

**Definition 1.1.** The generalized evaluation subgroups \( G_n^f(X, A, \ast) \) are defined by

\[
G_n^f(X, A, \ast) = \text{Im}(\omega_* : \pi_n(X^A, f) \to \pi_n(X, \ast)),
\]

where \( \omega : X^A \to X \) is the evaluation map from \( X^A \) to \( X \) and \( f : A \to X \) is a pointed (based) map.

\( G_n^f(X, A, \ast) \) consists of all elements \( \alpha \in \pi_n(X, \ast) \) such that there is a map \( F : A \times S^s \to X \) with \( F|_A = f, [F|_{S^s}] = \alpha \). This definition is a generalization of the definition of \( G_n(X) \).

Before continuing the next section we will prove some theorems concerning \( G_n^f(X, A, \ast) \) for a simple space \( X \).

**Theorem 1.2.** Let \( X \) be simple. If \( f, g : (A, \ast) \to (X, \ast) \) are homotopic, then \( G_n^f(X, A, \ast) = G_n^g(X, A, \ast) \).

**Proof.** Let \( H : A \times I \to X \) be the homotopy from \( f \) to \( g \). Since \( A \) is locally compact and Hausdorff, we have an adjointed map \( \sigma = \phi(H) : I \to X^A \). Thus \( \sigma \) is a path from \( f \) to \( g \) in \( X^A \). Consequently \( \sigma \) induces an isomorphism

\[
\sigma_* : \pi_n(X^A, g) \cong \pi_n(X^A, f).
\]

Moreover we have a commutative diagram

\[
\begin{array}{ccc}
\pi_n(X^A, g) & \xrightarrow{\sigma_*} & \pi_n(X^A, f) \\
\omega_* \downarrow & & \downarrow \omega_* \\
\pi_n(X, \ast) & \xrightarrow{(\omega \sigma)_*} & \pi_n(X, \ast).
\end{array}
\]
Since $X$ is simple, $(\omega \sigma)_*$ acts simply on $\pi_n(X, *)$.

Thus we have

\[
G_n^*(X, A, *) = (\omega \sigma)_*(G_n^*(X, A, *))
\]
\[
= (\omega \sigma)_*(\omega_*(\pi_n(X^A, g)))
\]
\[
= \omega_*\sigma_*(\pi_n(X^A, g))
\]
\[
= \omega_*(\pi_n(X^A, f))
\]
\[
= G_n^f(X, A, *).
\]

**THEOREM 1.3.** If $h: X \rightarrow Y$ and $f: A \rightarrow X$ are pointed maps, then $h_*: \pi_n(X, *) \rightarrow \pi_n(Y, *)$ carries $G_n^f(X, A, *)$ into $G_n^{hf}(Y, A, *)$.

**Proof.** For $\alpha \in G_n^f(X, A, *)$, there exists $F: A \times S^n \rightarrow X$ such that $[F|_{S^n}] = \alpha$, $F|_A = f$.

Define $H: A \times S^n \rightarrow Y$ by the composition

\[
A \times S^n \xrightarrow{F} X \xrightarrow{h} Y.
\]

Then $H$ is an affiliated map to $h_*(\alpha)$. This completes the proof.

**COROLLARY 1.4.** Let $X$ be simple and $f: A \rightarrow X$ be a pointed map. If $h: (X, *) \rightarrow (Y, *)$ is a homotopy equivalence, then $h_* : G_n^f(X, A, *) \cong G_n^{hf}(Y, A, *)$.

**Proof.** Since $h^{-1}hf$ and $f$ are homotopic, we have $G_n^{h^{-1}hf}(X, A, *) = G_n^f(X, A, *)$, where $h^{-1}$ is a homotopy inverse of $h$. Since $h_* : \pi_n(X, *) \rightarrow \pi_n(Y, *)$ is an isomorphism, it suffices to show that $h_* : G_n^f(X, A, *) \rightarrow G_n^{hf}(Y, A, *)$ is onto.

So that $h_*$ is an isomorphism.

**THEOREM 1.5.** Let $k: A \rightarrow B$ and $f: B \rightarrow X$ be pointed maps. Then $G_n^f(X, B, *) \subseteq G_n^{fk}(X, A, *)$.

**Proof.** For $\alpha \in G_n^f(X, B, *)$, there is a map such that $[F|_{S^n}] = \alpha$, $F|_B = f$.

Define $H: A \times S^n \rightarrow X$ by the composite

\[
A \times S^n \xrightarrow{k \times 1} B \times S^n \xrightarrow{F} X.
\]
Then $H$ is an affiliated map to $\alpha$ with respect to $A$.

**Corollary 1.6.** Let $X$ be simple and $f : B \to X$ be a pointed map. If $k : (A, *) \to (B, *)$ is a homotopy equivalence, then $G_n^f(X, B, *) = G_n^{fk}(X, A, *)$.

**Proof.** Since $fkk^{-1}$ and $f$ are homotopic, we have $G_n^f(X, B, *) \subseteq G_n^{fk}(X, A, *) \subseteq G_n^{fkk^{-1}}(X, B, *) = G_n^f(X, B, *)$.

2. Localizations and $G_n^f(X, A, *)$

Let $e_p : X \to X_p$ be the $p$-localizing map. Then $e_p$ induces the $p$-localizing homomorphism

$$e_p : \pi_n(X, *) \to \pi_n(X_p, *)$$

If $\alpha \in G_n^f(X, A, *)$, we then have $e_p^*(\alpha) \in G_n^{e_pf}(X_p, A, *)$ by Theorem 1.3.

Conversely, we have

**Theorem 2.1.** If $e_p^*(\alpha) \in G_n^{e_pf}(X_p, A, *)$ for all prime $p$ and 0, then $\alpha \in G_n^f(X, A, *)$.

**Proof.** Case 1. $\alpha$ is of finite order. Let $p_1^{\omega_1} p_2^{\omega_2} \cdots p_n^{\omega_n}$ be the prime factorization of the order of $\alpha$. Let $P_1 = \prod p_i^{\omega_i}$. Since $p_i^{\omega_i}$ and $P_1$ are relatively prime, there are integers $r$ and $s$ such that $1 = rp_1^{\omega_1} + sP_1 : so \alpha = r\alpha + sP_1 \alpha$. $sP_1 \alpha$ is of order $p_1^{\omega_1}$ while $r\alpha$ is of order $P_1$.

$r\alpha$ can then be written as a sum of its (hence's) multiples such that one summand is of order $p_2^{\omega_2}$ and the other of order $\prod_{i \geq 2} p_i^{\omega_i}$.

By induction $\alpha = k_1 \alpha + k_2 \alpha + \cdots + k_n \alpha$ where $k_i \alpha$ is of order $p_i^{\omega_i}$. Since $e_p^*(\alpha) \in G_n^{e_pf}(X_p, A, *)$ for all $p$, $e_p^*(k_i \alpha) \in G_n^{e_pf}(X_p, A, *)$. So if we show the results for $\alpha$ of order $p^m$, $p$ any prime, then each $k_i \alpha \in G_n^f(X, A, *)$ and thus $\alpha \in G_n^f(X, A, *)$.

Let $\alpha$ be of order $p_1^{m_1}, p$ prime. Localize at this $p$. Consider the following commutative diagram

$$\begin{array}{ccc}
\pi_n(X, f) & \xrightarrow{(\omega)_*} & \pi_n(X, *) \\
e_p^* \downarrow & & \downarrow e_p^* \\
\pi_n(X_p, e_p f) & \xrightarrow{(\omega)_*} & \pi_n(X_p, *) = \pi_n(X, *) \otimes Q_p
\end{array}$$
and the definition of $e_p\ast\alpha \in G_{n^*}(X_p, A, \ast)$, we then have $\tilde{\alpha} \in \pi_n(X, A, e_p f)$ such that $\omega_\ast(\tilde{\alpha}) = e_p\ast\alpha$.

Since $\hat{\pi}_p : \pi_n(X, f) \to \pi_n(X_p, e_p f)$ is a $p$–isomorphism, there exist $x \in \pi_n(X, f)$ and integer $q$ prime to $p$ such that $\hat{\pi}_p\ast(x) = \tilde{\alpha}^q$. Thus we have $\omega_\ast(x) \otimes 1 = e_p\ast\omega_\ast(x) = \omega_\ast(\tilde{\alpha}^q) = (\omega_\ast(\tilde{\alpha}))^q = (e_p\ast(\tilde{\alpha}))^q = q(\tilde{\alpha} \otimes 1) = q\alpha \otimes 1$. Hence $\omega_\ast(x) = q\alpha + \gamma$ where $\gamma$ has order $q'$ prime to $p$. Consequently, $\omega_\ast(q'x) = q'q\alpha$. Since $(q'q, p^n) = 1$, there are integers $r$ and $s$ such that $rp^n + sq'q = 1$. Thus $\omega_\ast(sq'x) = sq'q\alpha = \alpha - rp^m\alpha = \alpha$. Since $sq'x$ is in $\pi_n(X, A, \ast)$, $\alpha \in G_{n^*}(X, A, \ast)$.

Case 2. $\alpha$ is of infinite order.

Localizing at $0$, we can similarly obtain that there is an element $x \in \pi_n(X, f)$ such that $\omega_\ast(q'x) = q'q\alpha$ for some non-zero integers $q'$, $q$. Thus there are non-zero multiples of $\alpha$ in $G_{n^*}(X, A, \ast)$. Let $\tilde{\alpha}$ be the least positive integer such that $\tilde{\alpha} \alpha \in G_{n^*}(X, A, \ast)$. If $\tilde{\alpha} = 1$, let $p$ be a prime factor of $\tilde{\alpha}$. Localizing at this $p$, we have

$$x' \in \pi_n(X, f), \quad q'' \text{ (prime to } p)$$

such that $\omega_\ast(x') = q''\alpha$. But $q''\alpha$ is in the subgroup generated by $\tilde{\alpha}\alpha$, so $q''$ is multiple of $\tilde{\alpha}$. This is a contradiction since $p$ is a factor of $\tilde{\alpha}$ but not of $q''$. Thus $\tilde{\alpha} = 1$ and $\alpha \in G_{n^*}(X, A, \ast)$.

In the proof, we also obtain the following

**Corollary 2.2.** If $e_p\ast\alpha \in G_{n^*}(X, A, \ast)$, then there is a $q$ prime to $p$ such that $q\alpha \in G_{n^*}(X, A, \ast)$.

**Corollary 2.3.** $G_{n^*}(X, A, \ast)_p \equiv G_{n^*}(X, A, \ast)$.

**Proof.** It suffices to show that the map

$$e_p\ast : G_{n^*}(X, A, \ast) \to G_{n^*}(X_p, A, \ast)$$

is a $p$–isomorphism and $G_{n^*}(X, A, \ast)$ is $p$–local. Since $e_p\ast : \pi_n(X, \ast)$ is a $p$–isomorphism, $e_p\ast(\pi_n(X, \ast))$ is clearly $p$–injective.

Now let $\alpha \in G_{n^*}(X, A, \ast)$, then there is $\tilde{\alpha} \in \pi_n(X, A, e_p f)$ such that $\omega_\ast(\tilde{\alpha}) = \alpha$. Since $\hat{\pi}_p\ast$ is a $p$–isomorphism there exist $\tilde{\alpha} \in \pi_n(X, f)$ and an integer $q$ prime to $p$ such that $\hat{\pi}_p\ast(\tilde{\alpha}) = \tilde{\alpha}^q$. Thus we have $\omega_\ast(\tilde{\alpha}^q) = (\omega_\ast(\tilde{\alpha}))^q = \alpha^q$. That is $e_p\ast(\omega_\ast(\tilde{\alpha})) = \alpha^q$. This implies $e_p\ast$ is $p$–surjective. Moreover, we can easily prove that $G_{n^*}(X, A, \ast)$ is $p$–local.
DEFINITION 2.1. A property $P$ of a space $X$ is said to be a local property if the following is true:

$X$ has $P$ iff $X_p$ has $P$, for each prime $p$.

COROLLARY 2.4. That $\omega_* : \pi_n(X^A) \rightarrow \pi_n(X)$ is an epimorphism is a local property.

COROLLARY 2.5. If $G_{\alpha}(X, A, *)$ is torsion without $p$-torsion, any fibration $\tilde{f} : E \rightarrow S^{n+1}$ with fibre $X_p$ has a cross-section.

Proof. Since $G_{\alpha}(X, *) \leq G_{\alpha}(X, A, *)$, $G_{\alpha}(X, *)$ is torsion without $p$-torsion. By the result of Lang [8J we have the required result.

Since $X$ is simple, we see that $X_p$ is simple. Thus we have

THEOREM 2.6. $G_{\alpha}^p(X_p, A, *) = G_{\alpha}^p(X_p, A_p, *)$.

Proof. For $[h] \in G_{\alpha}^p(X_p, A, *)$, there is a map $F : A \times S^n \rightarrow X_p$ such that $F|_{s^n} = h, F|_{A} = e_p f$.

Consider the following commutative diagram obtained by localization,

\[
\begin{array}{ccc}
A \times S^n & \xrightarrow{F} & X_p \\
\downarrow{e_p \times 1} & & \downarrow{F_p} \\
A_p \times S^n & \xrightarrow{1 \times e_p'} & A_p \times S^n_p
\end{array}
\]

Define $F' : A_p \times S^n \rightarrow X_p$ by the composite

\[
A_p \times S^n \xrightarrow{1 \times e_p'} A_p \times S^n_p \xrightarrow{F_p} X_p.
\]

Then $F'|_{S^n} = F_p|_{S^n_p} \circ e_p' = F|_{S^n} = h$, $F'|_{A_p} = F_p|_{A_p}$. This shows that $[h] \in G_{\alpha}^p(A_p, X_p, A_p, *)$.
commutes we have $F_p|_{A_p} \sim f_p$. By Theorem 1.2 we obtain $[h] \in G_n f_p(X_p, A_p, \ast)$.

Conversely, if $[h] \in G_n f_p(X_p, A_p, \ast)$, then there is a map

$$F : A_p \times S^n \longrightarrow X_p$$

such that

$$F|_{A_p} = f_p, \quad F|_{S^n} = h.$$ Let $F' : A \times S^n \longrightarrow X_p$ be the composite

$$A \times S^n \xrightarrow{e_p \times 1} A_p \times S^n \xrightarrow{F} X_p.$$ Then $F'|_{S^n} = h, \quad F'|_{A} = f_p e_p = e_p f$.

By Theorem 2.6 we can rewrite

**Theorem 2.0.** If $\alpha \in G_n f(X, A)$, then $e_p\ast(\alpha) \in G_n f_p(X_p, A_p)$.

**Theorem 2.1.** If $e_p\ast(\alpha) \in G_n f_p(X_p, A_p)$ for all prime $p$ and 0, then $\alpha \in G_n f(X, A)$.

**Corollary 2.2.** If $e_p\ast(\alpha) \in G_n f_p(X_p, A_p)$, then there is an integer $q$ prime to $p$ such that $q \alpha \in G_n f(X, A)$.

**Corollary 2.3.** $G_n f(X, A) \leq G_n f_p(X_p, A_p)$.

**Remark.** These theorems are a generalization of the results in [8].

### 3. $G_n f(X, A, \ast)$ and $G(f)$

Mislin [9] introduced the term *genus* $G(X)$ to describe a set of homotopy type $X, Y, \ldots$ such that

$$X_p \simeq Y_p$$

for all prime $p$.

This definition is a motivation of the definition of $G(f)$.

**Definition 3.1 [7].** Let $f : X \longrightarrow Y$ be a map. The elements $[f'] \in G(f)$ ($f' : X' \longrightarrow Y'$) are equivalence classes of homotopy classes of $f'$ which satisfy:

For every prime $p$, there exist homotopy equivalences $h_p : X'_p \longrightarrow X_p$ and $k_p : Y'_p \longrightarrow Y_p$ so that $f_p h_p \sim k_p f'_p$.

(We denote the *genus* of $f$ either by $G(f)$ or by $G(X, Y, f)$.) The elements $[f'] \in G^X(f)$ ($f' : X \longrightarrow Y'$) are equivalence classes of hom-
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For every prime $p$, there exists a homotopy equivalence $k_p : Y'_p \xrightarrow{\sim} Y_p$ so that $k_p f'_p \sim f_p$. (Two maps $f_i : X \rightarrow Y_i$, $i=1, 2$ are equivalent under $X$ if there exists a homotopy equivalence $k : Y_1 \rightarrow Y_2$ with $kf_1 \sim f_2$.)

Similarly we can define $G_Y(f)$.

Assume that all spaces are pointed, simple, connected, finite CW-complexes and all maps are pointed.

**Theorem 3.1.** For $[f'] \in G^A(f) \left(f' : A \rightarrow Y\right)$ and $f : A \rightarrow X$, we have $G_n^f(X, A) \cong G_n^f(Y, A)$.

**Proof.** For each prime $p$, there exists a homotopy equivalence $k_p : Y'_p \rightarrow X_p$ so that $k_p f'_p \sim f_p'$.

Thus $k_p f'_p e_p \sim f_p e_p$, i.e., $k_p e_p f \sim e_p f$.

Consequently we have by Theorem 1.2 and Corollary 1.4

$$G_n^f(X, A)_p \cong G_n^f(X_p, A)$$

$$\cong G_n^f(X_p, A)$$

$$\cong G_n^f(Y_p, A)$$

$$\cong G_n^f(Y, A)_p$$

for all $p$.

Since $G_n^f(X, A)$ and $G_n^f(Y, A)$ are finitely generated abelian groups, Proposition 3.3 [5] completes the proof.

**Theorem 3.2.** If for $f : A \rightarrow X$, $[f'] \in G^X(f) \left(f' : B \rightarrow X\right)$, then $G_n^f(X, A) \cong G_n^f'(X, B)$.

**Proof.** By definition, there exists a homotopy equivalence $k_p : B_p \rightarrow A_p$ such that $f_p k_p \sim f_p'$. Thus we have by Theorem 1.2 and Corollary 1.6

$$G_n^f(X, A)_p \cong G_n^f(X_p, A_p)$$

$$\cong G_n^f(X_p, B_p)$$

$$\cong G_n^f(X, B)_p$$

for all $p$.

This completes the proof.

Combining Theorem 3.1 and 3.2, we have

**Theorem 3.3.** If for $f : A \rightarrow X$, $[f'] \in G(f) \left(f' : B \rightarrow Y\right)$,
then $G_n^f(X, A) \cong G_n^{f'}(Y, B)$.

**DEFINITION 3.2.** Let $A$ and $B$ be groups. We say that $A, B$ have same genus if $A_p \cong B_p$ for all prime $p$. Denote $B \in G(A)$.

From Theorem 3.3 we have

**COROLLARY 3.4.** If $Y \in G(X)$, $[g] \in G(f)(g : B \to Y)$ for $f : A \to X$. Then $G_n^g(Y, B) \in G(G_n^f(X, A))$ for all $n$. Thus we can assign to $G(f)$ a sequence $\{G(G_n^f(X, A))\}$.

**References**


Kookmin University,
Seoul 132, Korea