ANTI-ININVARIANT SUBMANIFOLDS OF LOCALLY CONFORMAL KÄHLER SPACE FORMS

Dedicated to Professor Chin Myung Chung on his sixtieth birthday

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The Hopf manifold is a typical example of a locally conformal Kähler manifold which admits no Kähler metric ([4]).

In this paper, we shall study anti-invànariant submanifolds of locally conformal Kähler manifolds and mainly prove the following theorems:

**THEOREM 1.** Let \( N^n \) be an \( n(>3) \)-dimensional anti-invariant submanifold of a locally conformal Kähler space form \( M^{2n}(H) \). If the associated vector field of the Lee form tangents to \( N^n \) and if the second fundamental tensors commute, then \( N^n \) is a conformally flat space.

**THEOREM 2.** Let \( N^n \) be an \( n(>3) \)-dimensional anti-invariant submanifold of a locally conformal Kähler space form \( M^{2n}(H) \) such that the associated vector field of the Lee form is normal to \( N^n \). If the second fundamental tensors commute, then \( N^n \) is a conformally flat space.

**THEOREM 3.** Let \( N^m \) be an \( m(>3) \)-dimensional anti-invariant submanifold of a locally conformal Kähler space form \( M^{2n}(H) \). If \( N^m \) is totally umbilical, then \( N^m \) is a conformally flat space.

1. Preliminaries

Let \( M^{2n}(F_{\mu}^4, g_{\mu}, \alpha_{\mu}) \) be an \( 2n \)-dimensional locally conformal Kähler manifold (an l.c.k-manifold). By its definition, at any point there exists a neighborhood in which a conformal metric \( g^e = e^{-2\rho} g \) is Kählerian, that is, \( \nabla^*(e^{-2\rho} F_\alpha^\mu) = 0 \), \( d\rho = \alpha \), where \( \nabla^* \) denotes the covariant

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differentiation with respect to $g^\ast$.

It is well known (cf. [1]) that a Hermitian manifold $M^{2n}(F_\mu, g_{\mu\lambda})$ is a l. c. k-manifold if and only if there exists a global 1-form $\alpha$ satisfying

\begin{align}
V_\nu F_{\mu\lambda} &= -\beta_\mu g_{\nu\lambda} + \beta_\lambda g_{\nu\mu} - \alpha_\mu F_{\nu\lambda} + \alpha_\lambda F_{\nu\mu}, \\
V_\mu \alpha_\lambda &= V_\lambda \alpha_\mu, \\
\beta_\lambda &= -\alpha_\mu F_{\lambda\mu}.
\end{align}

It is called a 1-form $\alpha$ the Lee form.

An l. c. k-manifold is called an l. c. k-space form if it has a constant holomorphic sectional curvature $H$. Then the Riemannian curvature tensor $R_{\mu\nu\rho\lambda}$ of an l. c. k-space form $M^{2n}(H)$ with constant holomorphic sectional curvature $H$ is given by (cf. [1])

\begin{align}
4R_{\mu\nu\rho\lambda} &= H (g_{\alpha\beta} g_{\nu\lambda} - g_{\alpha\rho} g_{\nu\lambda} + F_{\alpha\beta} F_{\nu\lambda} - F_{\alpha\rho} F_{\nu\lambda} - 2 F_{\alpha\nu} F_{\rho\lambda}) \\
&\quad + 3 (P_{\alpha\beta} g_{\nu\lambda} - P_{\alpha\rho} g_{\nu\lambda} + g_{\alpha\lambda} P_{\nu\beta} - g_{\alpha\rho} P_{\nu\beta}) - P_{\nu\beta} F_{\rho\lambda} \\
&\quad + P_{\alpha\lambda} F_{\nu\beta} - F_{\alpha\lambda} P_{\nu\beta} + F_{\alpha\beta} P_{\nu\lambda} - P_{\alpha\beta} F_{\nu\lambda} + 2 (P_{\nu\beta} F_{\rho\lambda} + F_{\nu\beta} F_{\rho\lambda}),
\end{align}

where

\begin{align}
P_{\mu\lambda} &= -V_\mu \alpha_\lambda - \alpha_\mu \alpha_\lambda + \frac{1}{2} \|\alpha\|^2 g_{\mu\lambda}, \\
P_{\mu\lambda} &= -P_{\mu\lambda} F_{\alpha\beta}.
\end{align}

2. Anti-invavitant submanifolds of l. c. k-manifolds

Let $N^m$ be an $m$-dimensional manifold immersed in a $2n$-dimensional l. c. k-manifold $M^{2n}(F_\mu, g_{\mu\lambda}, \alpha_\lambda)$. Since the discussion is local, we may assume, if it is necessary, that $N^m$ is imbeded in $M^{2n}$. If the manifold $M^{2n}$ is covered by a system of coordinate neighborhood $\{\mathcal{U}, y^i\}$ and $N^m$ is covered by a system of coordinate neighborhoods $\{U, x^i\}$, where, here and in the sequel the indices $\kappa, \nu, \mu, \lambda, \ldots; k, j, i, h, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$; $\{1, 2, \ldots, m\}$ respectively, then the submanifold $N^m$ can be represented by $y^i = y^i(x^i)$. Here and in the sequel we identify vector fields in $N^m$ with the images under the differential mapping.

We put

\begin{align}
B_i \equiv \partial_i y^k \quad (\partial_i = \partial / \partial x^i)
\end{align}

and denote by $C_j$ $2n-m$ mutually orthogonal unit vectors normal to $N^m$, where here and in the sequel the indices $x, y, z, \ldots$ run over the range $\{1, 2, \ldots, 2n-m\}$. Then the metric tensor $g_{ij}$ of $N^m$ and that of
normal bundle are respectively given by
\[ g_{ji} = g_{\mu \lambda} B_{ji}^{\mu \lambda}, \quad g_{yx} = g_{\mu \lambda} C_{yx}^{\mu \lambda}, \]
where \( B_{ji}^{\mu \lambda} = B_{j}^{\mu} B_{i}^{\lambda} \) and \( C_{yx}^{\mu \lambda} = C_{y}^{\mu} C_{x}^{\lambda}. \)

If the transform by \( F_{x}^{i} \) of any vector tangent to \( N^{m} \) is orthogonal to \( N^{n} \), we say that the submanifold \( N^{m} \) is anti-invariant in \( M^{2n} \). Since the rank of \( F_{x}^{i} \) is \( 2n \), we have \( m \leq n \).

For an anti-invariant submanifold \( N^{m} \) in \( M^{2n} \), we have equations of the form
\begin{align*}
(2.2) & \\ (2.3) & \\ (2.4) & \\ (2.5)
\end{align*}
where \( B_{ij}^{\lambda} = B_{j}^{\mu} g_{\mu \lambda} \) and \( C_{x}^{\lambda} = C_{y}^{\mu} g_{\mu \lambda}. \)

Using \( F_{\mu \lambda} = -F_{\lambda \mu}, F_{\mu \lambda} = F_{x}^{\epsilon} g_{x}, \) we have from (2.2) and (2.3),
\begin{align*}
(2.6)
\end{align*}
where \( f_{ij} = f_{i} g_{xy}, f_{yi} = f_{y} g_{ji} \) and \( f_{yx} = f_{y} g_{xz}. \)

Applying \( F \) to (2.2)-(2.5) and using (1.3) and these equations, we find
\begin{align*}
(2.7)
\end{align*}
Differentiating (2.2)-(2.4) covariantly along \( N^{m} \) and using (1.1), (1.2), (1.3), (2.7), equations of Gauss
\begin{align*}
(2.8)
\end{align*}
and those of Weingarten
\begin{align*}
(2.9)
\end{align*}
where \( V_{j} \) denotes the operator of covariant differentiation along \( N^{m} \) and \( h_{ji}^{x} \) and \( h_{j}^{i} = h_{j}^{x} g^{x} g_{xy}, (g^{x}) = (g_{x})^{-1}, \) are the second fundamental tensors of \( N^{m} \) with respect to the normals \( C_{x}^{\epsilon} \), we find
\begin{align*}
(2.10)
\end{align*}
where \( \alpha^i = \alpha_j g^{ji} \), \( \beta^i = \beta_j g^{ji} \), \( \alpha^z = \alpha_x g^{zx} \) and \( \beta^z = \beta_x g^{zx} \).

On the other hand, the equations of Gauss, Codazzi and Ricci are respectively given by

(2.11) \[ R_{kjih} = R_{jipq} B_{kji}^{pq} + h_k h_i h_j h_i - h_i h_k h_j h_i, \]
(2.12) \[ R_{jipq} B_{kji}^{pq} C_{ij}^k = V_{kji} - V_{jik}, \]
(2.13) \[ R_{kjyx} = R_{jipa} B_{kjx}^{ipa} C_{xy}^k - (h_k^{ij} h_{jix} - h_j^{ij} h_{kix}), \]

where \( R_{kjih} \) and \( R_{kjyx} \) are covariant components of the curvature tensors of \( N^m \) and the normal bundle respectively, \( B_{kji}^{pq} = B^w_{kj} B^u_{ji} B^v_x \) and \( B_{kjx}^{ipa} = B^w_{kj} B^u_{ji} B^v_x \).

Let \( N^m \) be an anti-invariant submanifold of an l.c.k-space form \( M^{2n}(H) \). Then by using (1.4), (2.2) and (2.3) we find that the equations (2.11) and (2.13) of Gauss and Ricci reduce to respectively

(2.14) \[ 4R_{kjih} = h(g_k h_i g_j - g_k i h_j) + 3(P_k h_i g_j - P_i h_k g_j + g_k h P_j i) - 4(h_k h_j h_i x - h_i x h_k h_j), \]
(2.15) \[ 4R_{kjyx} = h(f_k x f_j y - f_k y f_j x) - \tilde{P}_{kx} f_j y + \tilde{P}_{ky} f_j x - f_k x \tilde{P}_{jy} + f_k y \tilde{P}_{jx} - 4(h_k x h_j x - h_j x h_k x) + 2 \tilde{P}_{kj} f_{yx}, \]

where we have put

(2.16) \[ P_{ji} = P_{ju} B_{j}^{iu}, \quad \tilde{P}_{jx} = \tilde{P}_{ju} B_{j}^{ux}, \quad \tilde{P}_{kj} = \tilde{P}_{ju} B_{kj}^{zu}. \]

3. Proof of Theorem 1

Let \( N^a \) be an \( n \)-dimensional anti-invariant submanifold of an l.c. k-space form \( M^{2n}(H) \). Then from (2.7), (i) and (iii), we can easily see that

(3.1) \[ f^y_x = 0. \]

Suppose that the associated vector field \( \alpha^x \) of the Lee form \( \alpha \) is tangent to \( N^a \), that is, \( \alpha^x = \alpha_y g^{yx} = 0 \). Then, from (2.7), (iv), (2.10), (iii) and (3.1), we have

(3.2) \[ V_j f_x^i = \delta_j^x f_x^i \alpha_h - f_{jx} \alpha^i. \]

Applying the operator \( V_k \) to (3.2) and using the Ricci identities, we have

\[-R_{kji} f_x^j + R_{kji} f_x^h = \delta^j_x (V_k f_x^h) \alpha_h - \delta^j_k (V_j f_x^h) \alpha_h + \delta^j_x V_k \alpha_h - \delta^j_k V_j \alpha_h - (V_k f_{jx} - V_j f_{kx}) \alpha^i - f_{jx} V_k \alpha^i + f_{kx} V_j \alpha^i, \]

from which, transvecting with \( f_{1x} \) and using (2.7) and (2.10) with \( \alpha_x = 0 \), we can easily obtain
(3.3) \[ R_{kji} = R_{kji}^{\gamma} f_{\gamma} f_{\gamma} + g_{ji}(F_{j} a_{i} + \alpha_{j} a_{i} - ||\alpha||^{2} g_{ii}) \]
\[ - g_{ki}(F_{j} a_{i} + \alpha_{j} a_{i} - ||\alpha||^{2} g_{ji}) + g_{ki}(F_{j} a_{i} + \alpha_{j} a_{i}) \]
\[ - g_{ji}(F_{k} a_{i} + \alpha_{k} a_{i}). \]

On the other hand, since \( \bar{P}_{kz} = P_{k} f_{z} \), (2.15) implies
\[ 4R_{kji} f_{z} f_{z} = H(g_{ki} g_{ji} - g_{kl} g_{lj}) - P_{k} g_{ji} + P_{ki} g_{ji} - g_{kl} P_{jl} \]
\[ + g_{ki} P_{ji} - 4(h_{k}^{\gamma} h_{j} h_{z} - h_{k}^{\gamma} h_{h} h_{x}) f_{\gamma} f_{z}, \]
which and (3.3) yield
\[ 4R_{kji} = (H + 3||\alpha||^{2})(g_{ki} g_{ji} - g_{kl} g_{lj}) \]
\[ + 3(g_{ji}(F_{j} a_{i} + \alpha_{j} a_{i}) - g_{ki}(F_{j} a_{i} + \alpha_{j} a_{i})) \]
\[ + g_{ki}(F_{j} a_{i} + \alpha_{j} a_{i}) - g_{ji}(F_{k} a_{i} + \alpha_{k} a_{i}) \]
\[ - 4(h_{k}^{\gamma} h_{j} h_{z} - h_{k}^{\gamma} h_{h} h_{x}) f_{\gamma} f_{z}, \]
because \( P_{ji} = - F_{j} a_{i} - \alpha_{j} a_{i} + \frac{1}{2} ||\alpha||^{2} g_{ji}. \) Hence we have
\[ (3.4) \quad R_{kji} = g_{ji} L_{ki} - g_{ki} L_{ji} + g_{kl} L_{ji} - g_{jl} L_{ki} \]
\[ - (h_{k}^{\gamma} h_{j} h_{z} - h_{k}^{\gamma} h_{h} h_{x}) f_{\gamma} f_{z}, \]
where \( L_{ji} = - \frac{1}{8}(H + 3||\alpha||^{2}) g_{ji} + \frac{3}{4} (F_{j} a_{i} + \alpha_{j} a_{i}). \)

If the second fundamental tensors commute, then from (3.4), we see that the submanifold \( N^{n} \) is conformally flat, provided that \( n > 3 \), which completes the proof of Theorem 1.

4. Proof of Theorem 2

Let \( N^{n} \) be an \( n \)-dimensional anti-invariant submanifold of an l.c.k-space form \( M^{2n}(H) \). Suppose that the associated vector field \( \alpha x \) of the Lee form \( \alpha \) is normal to \( N^{n} \), that is, \( \alpha x = \alpha g^{ij} = 0 \). Then, from (2.7), (v), (2.10), (iii) and (3.1), we have
\[ F_{j} f_{x} = 0, \]
which and the Ricci identities yield
\[ R_{kji} f_{x} f_{x}^{\gamma} = R_{k} f_{x} f_{x}^{\gamma}, \]
and consequently
\[ (4.1) \quad R_{kji} = R_{kji} f_{x} f_{x}^{\gamma}. \]

On the other hand, since \( \bar{P}_{kz} = P_{k} f_{z} \) and
\[ P_{k} = h_{k}^{\gamma} a_{x} + \frac{1}{2} ||\alpha||^{2} g_{kk}, \]
(2.15) and (4.1) imply
or equivalently

\begin{equation}
4R_{kji} = g_{kli}L_{ji} + g_{jli}L_{ki} - g_{kli}L_{ji} - g_{jli}L_{ki} - 4(h_j^k h_{xh} - h_j^k h_{kh} f_{j} f_{x}^i).
\end{equation}

If the second fundamental tensors commute, then (4.2) gives that the submanifold \( N^m \) is conformally flat, provided that \( n \geq 3 \). Thus we have Theorem 2.

5. Proof of Theorem 3

Let \( N^m \) be an \( m \)-dimensional anti-invariant submanifold of an l.c. \( k \)-space form \( M^{2n}(H) \). Suppose that the submanifold \( N^m \) is totally umbilical, that is

\[ h_{ji}^x = h^x g_{ji}, \quad h^x = \frac{1}{m} g_{ji} h_{ji}^x. \]

Then the equation (2.14) of Gauss implies

\[ 4R_{kji} = g_{kli}L_{ji} + g_{jli}L_{ki} - g_{kli}L_{ji} - g_{jli}L_{ki}, \]

where \( L_{ji} = (2h^x h_x - \frac{1}{2} H) g_{ji} + 3P_{ji} \). Hence \( N^m \) is conformally flat, provided \( m \geq 3 \), which completes the proof of Theorem 3.

References