CONVOLUTIONS OF MEASURES ATTRACTED TO STABLE MEASURES ON BANACH SPACES

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1. Introduction and Notation

Let $B$ and $B^*$ denote a real separable Banach space with the norm $\| \cdot \|$ and its topological dual, respectively. By $\langle x, x' \rangle$ we shall denote the dual pairing between $B$ and $B^*$. By a probability (prob.) measure on $B$ we shall always mean that it is defined on $\mathcal{B}(B)$, i.e. the smallest $\sigma$-algebra containing all the open sets (in norm topology) of $B$. If $\mu$ and $\nu$ are two prob. measures on $B$, the convolution of $\mu$ and $\nu$ is defined by

$$\mu \ast \nu (E) = \int_B \mu (E - x) d\nu (x)$$

for every $E \in \mathcal{B}(B)$. The symbol $\mu \ast n$ will denote $\mu$ convoluted $n$ times with itself. For $x \in B$, $\delta (x)$ will denote degenerate prob. measure concentrated at the point $x$. By $\mathbb{R}$ and $\mathbb{R}^+$ we shall denote the set of real numbers and positive real numbers, respectively. For any $a \in \mathbb{R}^+$, $T_a \mu$ is defined to be the prob. measure on $B$ given by $T_a \mu (E) = \mu (a^{-1} E)$ for every $E \in \mathcal{B}(B)$. The characteristic functional of $\mu$ is defined by the formula

$$\hat{\mu} (x') = \int_B e^{i \langle x, x' \rangle} d\mu (x)$$

for every $x' \in B^*$. It is well known that the characteristic functional determines uniquely the prob. measure.

We say that $\mu$ is stable (prob.) measure of index $\alpha$ $(0 < \alpha \leq 2)$ on $B$ if for any $a, b \in \mathbb{R}^+$, there exist a number $c \in \mathbb{R}^+$ with $c^\alpha = (a^\alpha + b^\alpha)$ and an element $x \in B$ such that

$$T_a \mu \ast T_b \mu = T_c \mu \ast \delta (x).$$

A sequence $\{\mu_n\}$ of prob. measures on $B$ is said to converge weakly a

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prob. measure \( \mu \) (in symbols, \( \lim_{n \to \infty} \mu_n = \mu \) or \( \mu_n \to \mu \)) on \( B \) if for every bounded continuous real valued function \( f \) on \( B \)

\[
\int_B f \, d\mu_n \to \int_B f \, d\mu.
\]

We say that a non-degenerate probability measure \( \nu \) on \( B \) belongs to the domain of attraction of a non-degenerate prob. measure \( \mu \) of index \( \alpha (0 < \alpha \leq 2) \), denoted by \( \nu \in D(\alpha) \) if there exist a sequence \( \{a_n\} \) in \( \mathbb{R}^+ \), called normalizing coefficients, and a sequence \( \{x_n\} \) in \( B \), called centering vectors such that

\[
\lim_{n \to \infty} T_{a_n} \nu * \delta(x_n) = \mu.
\]

It is known in [3] that stable measures and only stable measures have nonempty domain of attraction.

The purpose of this paper is to show that if \( \lambda \in D(\alpha) \) and \( \nu \in D(\beta) \) where \( 0 < \alpha \leq \beta \leq 2 \), and if \( \{a_n\} \) and \( \{b_n\} \) are normalizing coefficients respectively of \( \lambda \) and \( \nu \), then \( \lambda \ast \nu \in D(\alpha) \) and its normalizing coefficients are \( \{(a_n^{-\alpha} + b_n^{-\alpha})^{-1/\alpha}\} \). This result extends a result, due to Tucker [7] proved in real line \( \mathbb{R} \) to a real separable Banach space setting.

2. Preliminaries

In this section we collect necessary definitions and some known results which will be needed in this paper.

**Definition 2.1.** A function \( V \) is said to be *regularly varying with exponent \( \gamma \) \((-\infty < \gamma < \infty)\) if it is real-valued, positive and measurable on \([a, \infty)\) for some \( a > 0 \), and if for each \( x > 0 \),

\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\gamma.
\]

A function \( L \) which is regularly varying with \( \gamma = 0 \) is called *slowly varying*.

It can be shown that \( U(\cdot) \) is regularly varying if and only if it can be written in the form

\[
U(x) = x^\gamma L(x)
\]

where \( -\infty < \gamma < \infty \) and \( L(\cdot) \) is slowly varying.

**Lemma 2.2** [5]. Let \( L, L_1 \) and \( L_2 \) denote slowly varying functions. Then the following holds
(1) for any $\delta > 0$, $x^{-\delta}L(x) \to 0$ as $x \to \infty$.

(2) $L^2$ for any $\delta \in \mathbb{R}$, $L_1 + L_2$, $L_1 \cdot L_2$ and $L_1/L_2$ are also slowly varying functions.

**Lemma 2.3** [2]. Let $\mu$ be a prob. measure on $B$. Then $\mu$ is stable of index $\alpha$ $(0 < \alpha \leq 2)$ if and only if either, for every $x' \in B^*$

(1) $\hat{\mu}(x') = \exp \{i \langle x_0, x' \rangle - \frac{1}{2} \langle Tx', x' \rangle \}$ if $\alpha = 2$

where $x_0 \in B$ and $T$ is the covariance operator from $X^*$ into $X$, or

(2) there exists a finite Borel measure $\Gamma$ on $S = \{ x \in B : \| x \| = 1 \}$ and a vector $x_0 \in B$ such that for every $x' \in B^*$

$$\hat{\mu}(x') = \exp \{i \langle x_0, x' \rangle - \int_S \langle s, x' \rangle d\Gamma(s) + iC(\alpha, x') \}$$

where

$$C(\alpha, x') = \begin{cases} \tan \frac{\pi \alpha}{2} \int_S \langle s, x' \rangle \langle s, x' \rangle^{-1} d\Gamma(s) & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \int_S \langle s, x' \rangle \log |\langle s, x' \rangle| d\Gamma(s) & \text{if } \alpha = 1. \end{cases}$$

If $\mu$ is a stable measure on $B$ of index $\alpha$, in view of Lemma 2.3, we shall denote it by $\mu = [2, x_0, T]$ when $\alpha = 2$, and by $\mu = [\alpha, x_0, \Gamma]$ when $0 < \alpha < 2$.

Let $\mu$ be an infinitely divisible prob. measure on $B$. Then it is known [6] that for any $t > 0$, there exists an infinitely divisible prob. measure $\nu$, denoted by $\mu'$, on $B$ such that $\hat{\nu}(x') = (\hat{\mu}(x'))^t$ for every $x' \in B^*$.

**Lemma 2.4** [1]. Let $\mu$ be an infinitely divisible prob. measure on $B$. Then $\mu$ is stable of index $\alpha$ $(0 < \alpha \leq 2)$ if and only if for any $t > 0$, there exists a vector $x_i \in B$ such that $\mu' = T_{t^{-1/n}} x_i$. 

3. Convolutions of measures in the domain of attraction of stable measures and their normalizing coefficients.

In this section we investigate the following problem: If $\lambda \in D(\alpha)$ and $\nu \in D(\beta)$ where $0 < \alpha \leq \beta \leq 2$, and if $\{a_n\}$ and $\{b_n\}$ are normalizing coefficients, then what can be said about the domain of attraction and the normalizing coefficients for $\lambda \ast \nu$? We first begin with a lemma.

**Lemma 3.1.** Let $\nu$ be in $D(\alpha)$ with normalizing coefficients $\{a_n\}$ and
centering vectors \( \{x_n\} \). Then there exists a slowly varying function \( L \) defined on \([1, \infty)\) such that \( a_n = n^{-1/\alpha}L(n) \) for every positive integer \( n \).

Proof. For any \( u \geq 1 \), we define \( b(u) = a_{[u]} \), where \([u]\) denotes the integral part of \( u \). Then for any \( t > 0 \) and \( x' \in B^* \), we have

\[
\hat{\varphi}_{[u]}(b(u)x') \cdot e^{i<\langle b(u)\rangle x_k, x'>} = (\hat{\varphi}_{[u]}(b(u)) \cdot b(u)) x_k',
\]

where \( x_k' = x_k - \frac{[u]}{u} b(u) x_k \). Since

\[
\lim_{u \to \infty} \hat{\varphi}_{[u]}(b(u)x') \cdot e^{i<\langle b(u)\rangle x_k, x'>} = \beta(x')
\]

for every \( x' \in B^* \) and \([ut]/[u] \to t \), it follows that for every \( x' \in B^* \),

\[
\lim_{u \to \infty} (\hat{\varphi}_{[u]}(b(u)x') \cdot e^{i<\langle b(u)\rangle x_k, x'>})/[ut]/[u] = \beta'(x').
\]

Hence by using the same arguments as in [1, p.214], there exist \( c(t) \in \mathbb{R}^+ \) and \( x_t \in B \) such that

\[
\frac{b(u)}{b(u)} \to c(t) \quad \text{and} \quad \mu = T_{c(\Omega)} \mu t * \delta(x_t).
\]

From this along with Lemma 2.4 we have \( c(t) = t^{-1/\alpha} (0 < \alpha \leq 2) \). This shows that \( b(\cdot) \) is a regularly varying function on \([1, \infty)\). Hence there exists a slowly varying function \( L \) on \([1, \infty)\) such that \( b(u) = u^{-1/\alpha}L(u) \). Therefore we have a slowly varying function \( L \) on \([1, \infty)\) such that \( a_n = n^{-1/\alpha}L(n) \) for every positive integers \( n \).

**Theorem 3.2.** Let \( \lambda \in D(\alpha) \) and \( \nu \in D(\beta) \), where \( 0 < \alpha < \beta \leq 2 \), and let \( \{a_n\} \) be normalizing coefficients for \( \lambda \). Then \( \lambda \ast \nu \in D(\alpha) \) and \( \{a_n\} \) are normalizing for \( \lambda \ast \nu \).

Proof. Since \( \lambda \in D(\alpha) \) and \( \nu \in D(\beta) \), there exist sequences \( \{a_n\} \) and \( \{b_n\} \) in \( \mathbb{R}^+ \), and sequences \( \{x_n\} \) and \( \{y_n\} \) in \( B \) for \( \lambda \) and \( \nu \), respectively such that

\[
\lim_{n \to \infty} T_{a_n} \lambda^{*n} * \delta(x_n), \quad \text{and} \quad \lim_{n \to \infty} T_{b_n} \nu^{*n} * \delta(y_n)
\]

are stable measures of indices \( \alpha \) and \( \beta \), respectively. Now let \( \mu = \lambda \ast \nu \) and \( z_n = x_n + (a_n/b_n)y_n \). Then, for each \( n \)

\[
T_{a_n} \mu^{*n} * \delta(z_n) = T_{a_n} \lambda^{*n} * T_{a_n} \nu^{*n} * \delta(x_n) * \delta((a_n/b_n)y_n) = T_{a_n} \lambda^{*n} * \delta(x_n) * T_{a_n/b_n} (T_{b_n} \nu^{*n} * \delta(y_n)). \tag{3.3}
\]
From Lemma 3.1 there exist slowly varying functions $L_1$ and $L_2$ on $[1, \infty)$ such that $a_n = n^{-1/\alpha}L_1(n)$ and $b_n = n^{-1/\beta}L_2(n)$ for every positive integers $n$. Hence we have

$$\frac{a_n}{b_n} = n^{(1/\beta) - (1/\alpha)} \cdot \frac{L_1(n)}{L_2(n)}.$$ 

Since $(1/\beta) - (1/\alpha) < 0$, it follows from Lemma 2.2 that we have $a_n/b_n \to 0$ as $n \to \infty$. From the equation (3.3) and the continuity of convolution ([4], p. 57), it follows that

$$\lim_{n \to \infty} T_{a_n} \mu^* \ast \delta(x_n) = \lim_{n \to \infty} T_{a_n} \lambda^* \ast \delta(x_n).$$

This shows that $\mu = \lambda \ast \nu \in D(\alpha)$ with normalizing coefficients $\{a_n\}$.

**Theorem 3.3.** If $\lambda \in D(\alpha)$ and $\nu \in D(\beta)$, where $0 < \alpha \leq \beta \leq 2$, and if $\{a_n\}$ and $\{b_n\}$ are normalizing coefficients for $\lambda$ and $\nu$, respectively, then $\lambda \ast \nu \in D(\alpha)$ and $\{(a_n^{-\alpha} + b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients for $\lambda \ast \nu$.

**Proof.** Case (i) $0 < \alpha < 2$. Since $a_n/b_n \to 0$ as $n \to \infty$ in the proof of Theorem 3.3 we have

$$a_n^{-\alpha} \sim a_n^{-\alpha} + b_n^{-\alpha}$$

where the notation $u_n \sim v_n$ means that $u_n/v_n \to 1$ as $n \to \infty$. Hence from Theorem 3.3 it follows that $\lambda \ast \nu \in D(\alpha)$ and $\{(a_n^{-\alpha} + b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients for $\lambda \ast \nu$. Case (ii) $0 < \alpha = \beta \leq 2$. Since $\lambda, \nu \in D(\alpha)$, we can choose the centering vectors $\{x_n\}$ and $\{y_n\}$ in $B$ such that

$$\lim_{n \to \infty} T_{a_n} \lambda^* \ast \delta(x_n) = \eta_1$$
$$\lim_{n \to \infty} T_{b_n} \nu^* \ast \delta(y_n) = \eta_2$$

are stable measures of index $\alpha$ with $x_0 = 0$ in the representation of Lemma 2.3. First, let $\eta_1$ and $\eta_2$ be stable measures of index 2 with the representations $[2, 0, T_1]$ and $[2, 0, T_2]$, respectively. Then for any $a, b \in \mathbb{R}^+$ $T = a^2 T_1 + b^2 T_2$ is a covariance operator from $B^*$ into $B$. Hence there exists a stable measure $\eta$ of index 2 with the representation $\eta = [2, 0, T]$. Next, let $\eta_1$ and $\eta_2$ be stable measures of index $\alpha (0 < \alpha < 2)$ with the representations $[\alpha, 0, \Gamma_1]$ and $[\alpha, 0, \Gamma_2]$, respectively. Then for any $a, b \in \mathbb{R}^+$ $\Gamma = a^\alpha \Gamma_1 + b^\alpha \Gamma_2$ is a finite measure on $S$. Hence it follows from Lemma 2.3 that there exists a stable measure $\eta$ of index $\alpha$ with the representation $\eta = [\alpha, 0, a^\alpha \Gamma_1 + b^\alpha \Gamma_2]$. Now for any $a, b \in \mathbb{R}^+$ let $\Gamma_3 = \frac{2}{\pi}((\alpha \log a) \Gamma_1 + (\alpha \log b) \Gamma_2)$. Since $\Gamma_1$ and $\Gamma_2$ are finite measure on $S$ and so
Thus for each $i=1,2$, there exists $y_i \in B$ such that
\[ y_i = B \int_S s d\Gamma_i(s) \]
where $B$ stands for the Bochner integral with respect to $\Gamma_i$ on $S$. Hence there exists $y_0 \in B$ such that for every $x' \in B^*$,
\[ \int_S \langle s, x' \rangle d\Gamma_2(s) = \langle y_0, x' \rangle. \]
In either case $\alpha=2$ or $0<\alpha<2$, for any $a, b \in \mathbb{R}^+$ there exists a stable measure $\eta$ of index $\alpha$ such that for any $x' \in B^*$
\[ (T_\alpha \eta_1 + T_b \eta_2)(x') = \hat{\eta}_1(\alpha x') \hat{\eta}_2(b x') = \begin{cases} \hat{\eta}(x') & \text{if } \alpha \neq 1 \\ \exp \{i \langle y_0, x' \rangle \} & \text{if } \alpha = 1. \end{cases} \]

Now let $L_1$ and $L_2$ be slowly varying functions on $[1, \infty)$ such that $a_n = n^{-1/\alpha} L_1(n)$ and $b_n = n^{-1/\alpha} L_2(n)$ for every positive integers $n$. Let's define a function $L$ on $[1, \infty)$ by
\[ L(x) = (L_1^{-\alpha}(x) + L_2^{-\alpha}(x))^{-\alpha}. \]
Then by Lemma 2.2, $L$ is a slowly varying function on $[1, \infty)$. Now let $\mu = \lambda * \nu$, and for each $n$ let $p_n = L(n)/L_1(n)$, $q_n = L(n)/L_2(n)$, $c_n = n^{-1/\alpha} L(n)$, and
\[ \mu_n = T_{c_n} \mu * \delta(p_n x_n + q_n y_n - h(\alpha)) \]
where $h(\alpha) = y_0$ if $\alpha = 1$, and $h(\alpha) = 0$ if $\alpha \neq 1$. Then we have, for each $n=1,2,...$, $p_n^\alpha + q_n^\alpha = 1$ and
\[ \mu_n = T_{p_n} (T_{a_n} \lambda^* n * \delta(x_n)) * T_{q_n} (T_{b_n} \nu^* n * \delta(y_n)) * \delta(-h(\alpha)). \] (3.4)
Let's write
\[ \lambda_n = T_{a_n} \lambda^* n * \delta(x_n), \ \nu_n = T_{b_n} \nu^* n * \delta(y_n). \]
Then we have $\mu_n = T_{p_n} \lambda_n * T_{q_n} \nu_n * \delta(-h(\alpha))$ for each $n=1,2,...$

We now shall show that $\mu_n \rightarrow \eta$ as $n \rightarrow \infty$. To show this, we first show that $\{\mu_n\}$ is conditionally compact. Let $\{\mu_m\}$ be a subsequence of $\{\mu_n\}$. Since $\{p_n\}$ is a bounded sequence in $\mathbb{R}^+$, the corresponding subsequence $\{p_m\}$ of $\{p_n\}$ to $\{\mu_m\}$ has a convergent subsequence $\{p_{m(k)}\}$ of $\{p_m\}$ and so the corresponding subsequence $\{q_{m(k)}\}$ of $\{q_m\}$ is also convergent. Thus there exists $a \geq 0$ and $b \geq 0$ such that $p_{m(k)} \rightarrow a$ and
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$q_m(k) \to b$ as $k \to \infty$. Hence from (3.4), we have

$$\lim_{k \to \infty} \mu_m(k) = T_\alpha \eta_1 \ast T_\beta \eta_2 \ast \delta(-h(\alpha)) = \eta.$$ 

This shows that $\{\mu_n\}$ is conditionally compact. We next show that $\mu_n(\cdot)$ converges to $\hat{\eta}(\cdot)$. Since

$$T_{p_m(k)}^{\hat{\lambda}_m(k)}(x') \to \hat{\eta}_1(ax'), \quad T_{q_m(k)}^{\hat{\lambda}_m(k)}(x') \to \hat{\eta}_2(bx')$$

for every $x' \in B^*$, we have

$$\hat{\mu}_m(k)(x') \to \hat{\eta}_1(ax') \hat{\eta}_2(bx') \exp\{-h(\alpha)\} = \hat{\eta}(x').$$

Thus the subsequence $\{\hat{\mu}_m(\cdot)\}$ of $\{\mu_n(\cdot)\}$ has further convergent subsequence $\{\hat{\mu}_m(\cdot)\}$ which converges to $\hat{\eta}(\cdot)$. Hence $\mu_n(\cdot) \to \hat{\eta}(\cdot)$ as $n \to \infty$. Consequently, by Lemma 2.1 ([4], p. 153), $\mu_n$ converges weakly to a stable measure $\eta$ of index $\alpha$. This shows that $\lambda * \nu \in D(\alpha)$ and $\{(a_n^{-\alpha} + b_n^{-\alpha})^{-1/\alpha}\}$ are normalizing coefficients of $\lambda * \nu$.

References


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