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An Application of Multipole Expansion to the Computation of Gravity Anomalies

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Abstract: The computation of gravity anomalies by multipole expansion is derived and compared with exact calculation for right rectangular prisms and right circular cylinders. For sources near field points, the multipole expansion results in a better approximation in volume integrals than in surface integrals. Nonetheless two approximate methods are coincident in the far-field of the general geophyical prospecting.

INTRODUCTION

The exact solution of the gravity anomalies for right rectangular prisms and right circular cylinders has been investigated by various authors (e.g. Kolbenheyer, 1962; Nabighian, 1962; Nagy 1965 and 1966). The multipole expansion of the approximate solution was studied by Tychonoff and Samarski (1959) for the mathematical development of the potential theory and Bodvarsson (1970) for the theoretical view of gravity. Nonetheless none have inspected and interpreted the difference between the exact solution and the approximate computation of the multipole expansion for the sake of the practical use in geomathematical problems. The objective of this work is to find the percent difference between the exact solution and the approximation of the multipole expansion.

The multipole expansion by volume integrals is found to be more accurate than that by surface integrals in the near-field. The gravity results, however, computed by two approximate methods are coincident in the far-field as the observation point gets away from the source of the body. The computation accuracy of gravity by approximate methods depends on the dimension of the body and the distance of the field point from the body.

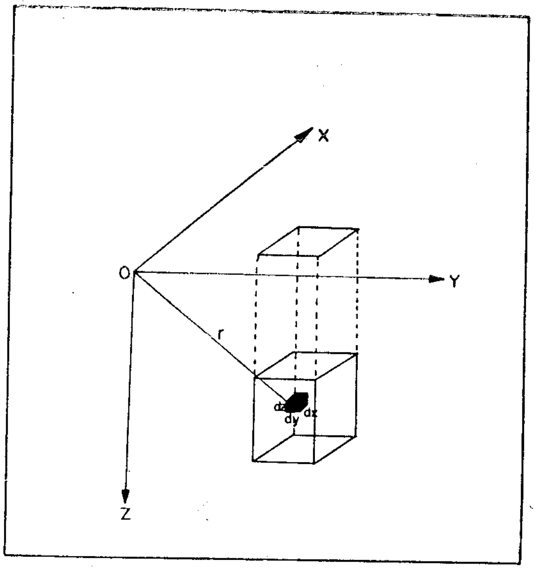

Fig. 1 A right rectangular prism and the Cartesian coordinate system.

An exact solution of gravity for a right rectangular prism is obtained from triple integrals. The exact solution of the gravity of a right circular cylinder is derived from the complete elliptical integrals of the first and the second kind and Heuman's Lamda function. The computing procedure for the exact solution is quite often complicated and laborious. Consequently the geophysical prospecting can take advantage

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of the simplified computation via the multipole expansion.

The second terms of the multipole expansion are exclusively important for the near-field, while they can be neglected for the far-field due to the large field distance of the potentials.

EXACT GRAVITY FIELD CALCULATION FOR VERTICAL PRISMS AND CYLINDERS

The mathematical details for a right rectangular prism is found in Nagy's (1965) work.

The gravity field of the volume element, ΔV is given by

$$\Delta F = G\rho \frac{\Delta V}{r^2} \quad (1)$$

Where G is the gravitational constant, ρ is the density contrast, and $r=\sqrt{x^2+y^2+z^2}$. The total vertical component of the field (see Figure 1) is

$$F_z = G\rho \int_V \frac{dV}{r^2}\cos\theta = G\rho \int \frac{z\,dV}{r^3} \quad (2)$$

Rewriting the above equation,

$$F_z = G\rho \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{z\,dz\,dy\,dx}{(x^2+y^2+z^2)^{3/2}} \quad (3)$$

After same arrangements,

$$F_z = G\rho \Bigg|_{x_1}^{x_2}\Bigg|_{y_1}^{y_2}\Bigg|_{z_1}^{z_2} x\ln(y+r) + y\ln(x+r) - z\arcsin\frac{z^2+y^2+yr}{(y+r)\sqrt{y^2+z^2}} \Bigg|\Bigg|\Bigg| \quad (4)$$

where $r=\sqrt{x^2+y^2+z^2}$

Letting $z_1=0$ and $z_2=h$,

$$\begin{aligned} F_z = {} & G\rho x_2 \ln\left(\frac{y_1+\sqrt{x_2^2+y_1^2+h^2}}{y_1+\sqrt{x_2^2+y_1^2}} \Big/ \frac{y_2+\sqrt{x_2^2+y_2^2+h^2}}{y_2+\sqrt{x_2^2+y_2^2}}\right) \\ & - G\rho x_1 \ln\left(\frac{y_1+\sqrt{x_1^2+y_1^2+h^2}}{y_1+\sqrt{x_1^2+y_1^2}} \Big/ \frac{y_2+\sqrt{x_1^2+y_2^2+h^2}}{y_2+\sqrt{x_1^2+y_2^2}}\right) \\ & + G\rho y_2 \ln\left(\frac{x_1+\sqrt{x_1^2+y_2^2+h^2}}{x_1+\sqrt{x_1^2+y_2^2}} \Big/ \frac{x_2+\sqrt{x_2^2+y_2^2+h^2}}{x_2+\sqrt{x_2^2+y_2^2}}\right) \\ & - G\rho y_1 \ln\left(\frac{x_1+\sqrt{x_1^2+y_1^2+h^2}}{x_1+\sqrt{x_1^2+y_1^2}} \Big/ \frac{x_2+\sqrt{x_2^2+y_1^2+h^2}}{x_2+\sqrt{x_2^2+y_1^2}}\right) \\ & + G\rho h \left(\arcsin\frac{h^2+y_2^2+y_2\sqrt{x_2^2+y_2^2+h^2}}{(y_2+\sqrt{x_1^2+y_2^2+h^2})\sqrt{y_2^2+h^2}} - \arcsin\frac{h^2+y_2^2+y_2\sqrt{x_1^2+y_2^2+h^2}}{(y_2+\sqrt{x_1^2+y_2^2+h^2})\sqrt{y_2^2+h^2}}\right. \\ & \left. - \arcsin\frac{h^2+y_2^2+y_1\sqrt{x_2^2+y_1^2+h^2}}{(y_1+\sqrt{x_2^2+y_1^2+h^2})\sqrt{y_1^2+h^2}} + \arcsin\frac{h^2+y_1^2+y_1\sqrt{x_1^2+y_1^2+h^2}}{(y_1+\sqrt{x_1^2+y_1^2+h^2})\sqrt{y_1^2+h^2}}\right) \end{aligned} \quad (5)$$

The coordinate system is taken as the rectangular Cartesian coordinate system for computation of the vertical gravity.

Exact expression of gravity of the right circular cylinder has been derived by Kolbenheyer (1962) and Nabighian (1962).

The vertical component of gravity at ρ which the depth of a_i is given by Fig. 2.

$$\Delta g_i = 2G\rho\left[\frac{1-x^2}{\sqrt{(1+x)^2+a^2}}K(k_i) + \sqrt{(1+x)^2+a^2}\,E(k_i) + \frac{\pi}{2}a\lambda_0(\varphi_i, k_i) - \pi a\right] \quad (6)$$

Where G is the gravitational constant ρ is the density

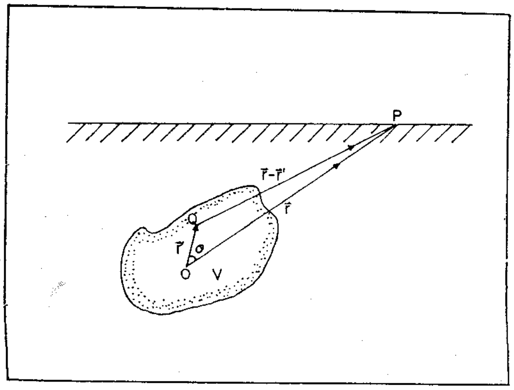

Fig. 2 A right circular cylinder and its parameters.

$$x=\frac{x_0}{R}$$

$$a=\frac{a_i}{R},\ i=1,2$$

K and E are the complete elliptical integrals of the first and the second kind with parameter

$$k_i=\frac{4x}{(1+x)^2+a^2}$$

$\lambda_6(\varphi_i, k_i)$ is Heuman's Lamda function

$$\varphi_i=\arcsin\frac{a}{\sqrt{(1-x)^2+a^2}}$$

The vertical gravity is obtained from subtraction of two depths

$$\Delta g=\Delta g_1-\Delta g_2 \qquad (7)$$

GRAVITY COMPUTATION BY MULTIPOLE EXPANSION

Gravity computation by multipole expansion is carried out by using volume integrals for finite structures and surface integrals for semi-infinite structures.

Finite structures (volume integrals)

This method has been discussed by Grant and West (1965). The gravity potential at any point outside a body (Fig. 3) is

$$u(\vec{r})=-G\int_V\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}|}dV' \qquad (8)$$

Where G is the gravitational constant and $\rho(\vec{r}')$ is the density throughout a volume V. It is worthwhile to find a simplified expression for equation (8). To do this we express the distance between a body point and the field point as

$$\frac{1}{|\vec{r}-\vec{r}_1|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}$$

$$=\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)\quad r'\leq r \qquad (9)$$

Where P_l is the Legendre polynomial of order l.

The gravity potential becomes

$$u(r)=-\frac{G}{r}\sum_{l=0}^{\infty}\int_V l(r')\left(\frac{r'}{r}\right)^l P_l(\cos\theta)dV' \qquad (10)$$

Grant and West (1965) have discussed this problem in detail.

Equation (10) can be rewritten as

$$u(r)=-\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\frac{4\pi G}{2l+1}\int_V l(r')(r')^l Y_l^{m*}(\theta_0,\varphi_0)dV'\frac{Y_l^m(\theta,\varphi)}{r^{l+1}}$$

$$=-\sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}b_l^m\frac{Y_l^m(\theta,\varphi)}{r^{l+1}} \qquad (11)$$

Where $b_l^m=\frac{4\pi G}{2l+1}\int_V\rho(r')(r')^l Y_l^{m*}(\theta_0,\varphi_0)dV'$

The b_l^m constitute multipole moments expressed in imaginary form. The multipole moments can be reduced to simple algebraic forms with a proper choice of the origin and axis of the coordinate system. In the coordinate system(X,

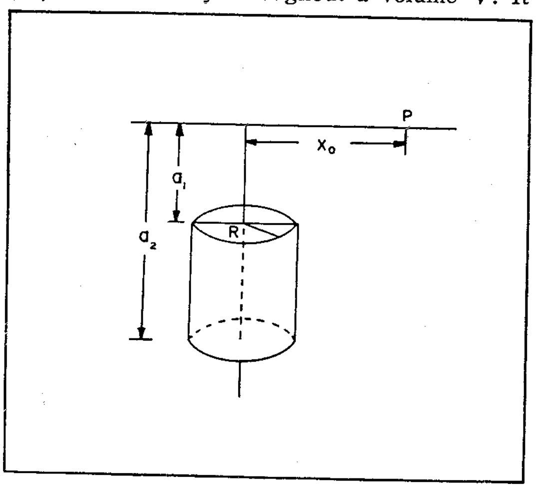

Fig. 3 The gravitational between the observation point P and a body V.

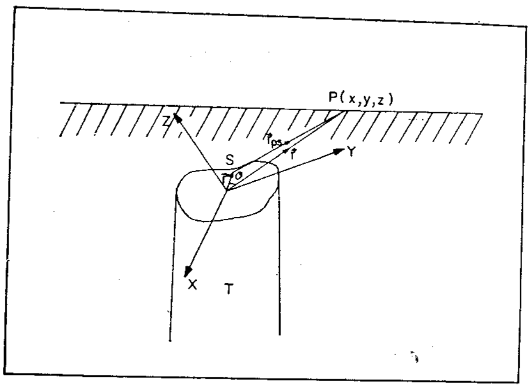

Fig. 4 Generalized coordinate for a vertical finite structure. A simplified model whose body and spatial axes coincide has been discussed in this study.

Y, Z) used here, the origin lies at the center of mass and the spatial axes coincide with the body axes α, β and γ (Fig. 4).

It is convenient to change Equation (11) into a real form since all the multipole moments are real

$$u(r) = -\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{B_l^m Y_l^m}{r^{l+1}} \qquad (12)$$

where

$$B_l^m Y_l^m = b_l^{m*} y_l^{m*} + b_l^m y_l^m$$

$$Y_l^m(\theta, \varphi) = \cos m\varphi P_l^{|m|}(\cos\theta)$$

$$B_l^m = (2-\delta_{l,m}) \frac{G(l-m)!}{(l+m)!} \int_V \rho(r')(r')^l \cos m\varphi P_l^{|m|}(\cos\theta_0)\, dV'$$

$$\delta_l^m = \begin{cases} 1 & l=m \\ 0 & l \neq m \end{cases}$$

Since equation (12) is an infinite number of multipole moments, they can hardly all be determined. Grant and West (1965) show that (12) is convergent. The first three terms of equation (12) are usually sufficient to approximate most gravimetric measurement. Taking into account the first three terms, equation (12) will be

$$-u(r) = \sum_{l=0}^{2} \sum_{m=0}^{l} \frac{B_l^m Y_e^m}{r^{l+1}} \qquad (13)$$

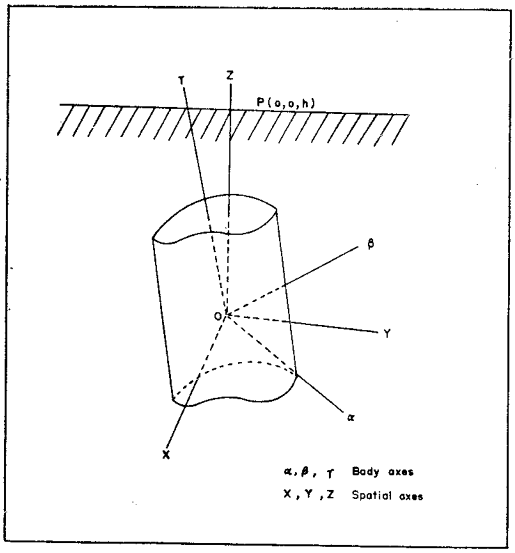

Fig. 5 The Cartesian coordinate system for a vertical semi-infinite structure.

The vertical component of gravity $\Delta g(p)$ is obtained by differentiating equation (13). The coefficients of $l=1$ and the coefficient B_2^1 vanish by symmetry. Therefore equation (13) becomes

$$-u(r) = \frac{B_0^0}{r} + \frac{B_2^0(3\gamma^2 - r^2)}{2r^5} + \frac{3B_2^2(\gamma^2 - \beta^2)}{r^5} \qquad (14)$$

where $r = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$

Finally the vertical component of gravity $\Delta g(p)$ is obtained by differentiating equation (14)

$$\Delta g(p) = \frac{B_0^0 \gamma}{r^3} - \frac{3B_2^0 \gamma}{2r^7}(3\alpha^2 + 3\beta^2 - 2\gamma^2) - \frac{15B_2^2 \gamma}{r^7}(\alpha^2 - \beta^2) \qquad (15)$$

Hence the vertical gravity at the depth of h as measured from the center of mass is

$$\Delta g(p) = \frac{B_0^0 h}{r^3} - \frac{3B_2^0 h}{2r^7}[3(\alpha^2 + \beta^2) - 2h^2] - \frac{15B_2^2 h}{r^7}(\alpha^2 - \beta^2) \qquad (16)$$

From eequation (16), the dimension and shape of the disturbing body may be estimated by the moments determined from the field measurements if the density of the homogeneous body is known.

Semi-infinite structures (surface integrals)

In potential theory (e.g., Kellogg, 1953), the volume integrals are usually used in the mathematical derivations. However, for practical cases, it is often convenient to transform volume integrals over homogeneous bodies into surface integrals. This transformation has been discussed by Tychonoff and Samarsky (1959) and Bodvarsson (1970).

The acceleration of gravity due to a finite body B at field point P is explained on Fig. 6.

$$\Delta g(p) = G \int_B \nabla_P \left(\frac{1}{r_{PQ}} \right) P(Q)\, dV_Q \qquad (17)$$

where G is the gravitational constant, r_{PQ} is the distance from P to the source point Q, and

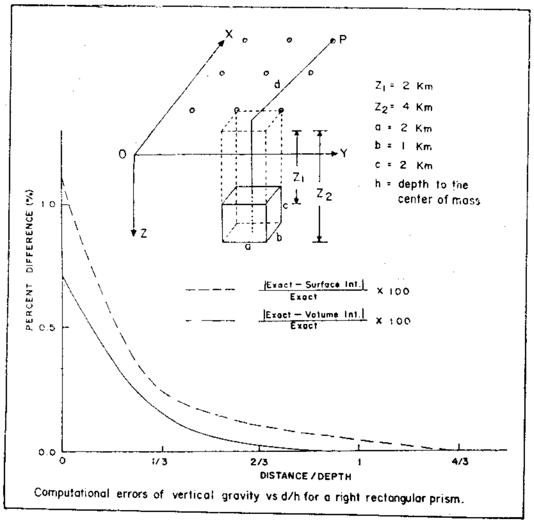

Fig. 6 Percent difference VS d/h for a right rectangular prism.

$\rho(Q)$ is the density of body B at Q. If the density ρ is constant inside body B, the density gradient can be expressed by the delta function.

$$\vec{\nabla}\rho(Q)=\rho\delta(Q-S)\vec{n} \tag{18}$$

where S is a point on the surface $\sum$ of the body B and $\vec{n}$ is the in ward normal vector to the surface $\sum$. Therefore

$$\Delta g(p)=G\rho\int_{B'}\frac{\delta(Q-S)\vec{n}(S)}{r_{PQ}}$$
$$=G\rho\int_{S}\frac{\vec{n}(S)da_S}{r_{PS}} \tag{19}$$

where B' is the bounded volume by the surface which completely encloses the body B and r_{PS} is the distance from the observation point to a point on the surface of the body.

The acceleration in the direction of the unit vector $\vec{k}$ is

$$\Delta g_k(P)=G\rho\int_{S}\frac{\vec{k}\cdot\vec{n}(S)}{r_{PS}}\,da_S \tag{20}$$

Equation (20) shows that the volume integral can be transformed into a surface integral on the condition that the density ρ is constant. Equation (20) can be used to calculate the vertical gravity $\Delta g(p)$ in case of structures bounded by vertical lateral faces such as the vertical rectangular prism and the vertical circular cylinder (e.g., vertical dikes and stocks). It is often possible to approximate complicated bodies in nature as simple bodies for geophysical work. From equation (21), since $\vec{k}\cdot\vec{n}=0$ on the vertical lateral faces, it is necessary only to compute gravity of the top and the bottom for finite bodies. As the gravity difference between the top and the bottom is calculated easily, this technique is suitable for a finite model. If the structures are semi-infinite or very deep structures, it is necessary only to compute gravity over the top. Thus the surface integral approximation is useful for semi-infinite bodies, while the volume integral method for finite bodies. It is convenient to assume that the origin of the coordinate system is located at the center of gravity of the surface of body T and that th spatial axes coincide with the principal axes of the body T.

$$\Delta g_k(p)=G\left[\frac{m}{r}+\frac{1}{r^3}(x\bar{x}m+y\bar{y}m+z\bar{z}m)\right.$$
$$+\frac{1}{2r^5}\Big\{\{(x^2-y^2)\ (M_1-M_2)+(y^2-z^2)$$
$$\left.(M_2-M_3)+(x^2-z^2)(M_1-M_3)\}\Big\}\right] \tag{21}$$

where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are coordinates of the center of gravity on the surface. M_1, M_2, and M_3 are the moments of inertia of the X-axis, Y-axis, and Z-axis of the surface of the body.

If the origin of the coordinates is located at the center of gravity on the surface and the coordinate axes coincide with the prinicipal axes of the moments of inertia, the gravity acceleration at the depth of Z_i is obtained by being truncated at the second order terms of the multipole expansion.

$$\Delta g_i(p)=G\left[\frac{m}{r_i}+\frac{1}{2r_i^5}\{(x^2-y^2)(M_1-M_2)\right.$$
$$+(y^2-z_i^2)(M_2-M_3)\ +(x^2-z_i^2)$$
$$\left.(M_1-M_3)\}\right] \tag{22}$$

$$r_i=\sqrt{x^2+y^2+z_i^2}$$

This asymptotic expression holds well when the distance r is large compared to the dimensions of the body. This approximate method is more convenient for infinite structures. For mathematical models, the right rectangular prism and the right circular cylinder with flat top and bottom are given. Assuming that the surface of the top and bottom of the body is flat, the mass of the sheet per unit area $c=\rho\vec{k}\cdot\vec{n}=\rho$, and $M_3=0$. The vertical gravity, therefore, is given by

$$\Delta g(p)=\Delta g_1-\Delta g_2 \qquad (23)$$

Convergence and Truncation of Series Expressions

Taking into the cosine law account,

$$\frac{1}{r_{PS}}=\frac{1}{r\sqrt{1-2\alpha\cos\theta+\alpha^2}}\ ,\quad \alpha=\frac{r'}{r}$$

$$=\frac{1}{r\sqrt{(1-\alpha e^{i\theta})(1-\alpha e^{-i\theta})}} \qquad (24)$$

The expansion in powers of α is absolutely convergent if $|\alpha e^{i\theta}|<1$ and $|\alpha e^{-i\theta}|<1$. Since $|e^{i\theta}|=|e^{-i\theta}|=1$ for all real value of θ, both conditions of equation (24) are satisfied when $|\alpha|<1$.

The expansion is absolutely and uniformly convergent if $|\alpha|<|\alpha_0|<1$. Hence the series expression in equation (24) for the potential can be integrated term by term. But it is too difficult to carry out the infinite integration. The simple and truncated expression provides a good and finite form for many practical uses. In this paper, the second and the fourth term vanish if the origin is located at the center of gravity or the center of mass and the body axes coincide with the spatial axes in the symmetrical bodies. The fourth term is small as compared to the whole expressions. Hence the first and the third term are mainly discussed in this work.

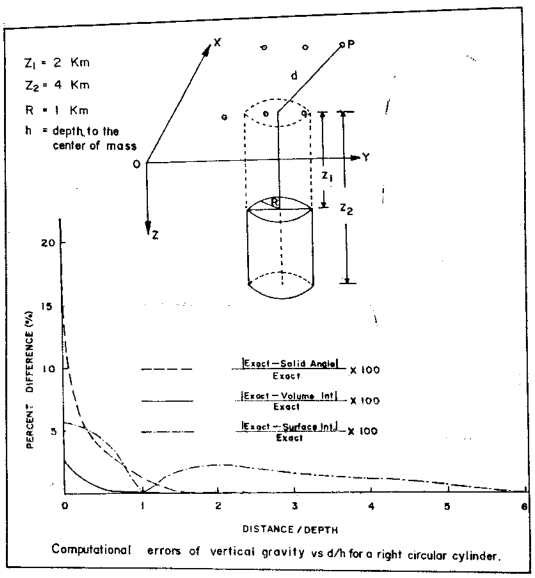

Fig. 7 Percent difference VS d/h for a right circular cylinder.

Gravity Computation for Right Rectangular Prisms and Right Circular Cylinders

The approximations from volume and surface integrals are computed by equations (16) and (23). It is assumed that the rectangular prism with a unit density has $a=1$km, $b=c=2$km, the depth to the center of the mass, $h=3$km, and that the cylinder has $R=1$km, and $h=3$ km with a unit density. For simplified calculations, the field point is moved from the origin ($x=y=0$) to Y-axis with $X=0$ with constant depth. The values of Tables 1 and 2 show the computational results comparing with the exact calculations. As shown on Fig. 6 and 7, these approximations converge to the exact values with increasing the distance of field point. Taking into account two integral methods, the volume integral yields a better approximation than the surface integral in the near-field. Nevertheless two methods are coincident Nettleton (1942) has discussed the solid angle method by using $g(p)=\Omega\times G\times\rho\times t$ where Ω is a solid angle, G is the gravitational constant, ρ is the density contrast, and t is the thickness of the cylinder. The solid angle values are obtained from Musket et al. (1956) as the field point distance is farther. The solid angle method for the right circular cylinder provides

Table 1 Numerical calculations for a right rectangular prism.

d/h	Exact Value (mgal)	Surface Inte. (mgal)	Volume Inte. (mgal)
0	3.066	3.031	3.088
1/6	2.947	2.930	2.960
1/3	2.629	2.635	2.626
1	1.073	1.074	1.073
4/3	0.650	0.650	0.650
5/3	0.408	0.408	0.408
2	0.267	0.267	0.267

d: field point distance from an observation point to the source point
h: depth to the center of mass

Table 2 Numerical calculations for a right circular cylinder.

d/h	Exact value (mgal)	Surface Inte. (mgal)	Volume Inte. (mgal)	Solid Angle (mgal)
0	14.932	11.787	14.487	14.077
1/3	13.602	12.998	13.518	12.894
2/3	10.293	10.622	10.348	9.968
1	6.758	6.861	6.768	6.748
4/3	4.217	4.228	4.216	4.292
2	1.740	1.737	1.740	1.780
8/3	0.833	0.831	0.833	0.847
3	0.605	0.605	0.605	0.614
10/3	0.452	0.452	0.452	0.458
4	0.270	0.270	0.270	0.273
5	0.142	0.142	0.142	0.143
6	0.083	0.083	0.083	0.084
20/3	0.061	0.061	0.061	0.061

considerable errors due to slow approach to the exact solution (Table 2).

DISCUSSION AND CONCLUSION

Approximation using the multipole expansion holds well when the distance to the field point is large as compared to the dimension of the body. The computation errors via the multipole expansion depend upon the relationship between the body dimension and the distance of the field point. The distance of the field point should be at least three times as large as the length of the body dimension, taking into account right rectangular prisms and right circular cylinders.

Gravity interpretation by the approximate methods is very useful for geophysical prospecting, since the exact calculation is very often complicated, time-consuming, and laborious. More-over, the approximate methods from potential theory are inverse problems that are often used to find out the shape and size of the anomalous bodies. The anomalous bodies could be estimated directly from the corresponding gravity anomalies observed along the surface. The anomalous bodies are also approximated as simple bodies such as prisms or cylinders. The approximation by surface integrals is good for the semi-infinite body, while the approximation by volume integrals is better for the finite body. According to the numerical calculation, the latter is more accurate than the former for geophysical prospecting in the near field.

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Multipole 확장에 의한 중력이상의 계산과 응용

김 소 구*

요약 : Multipole 확장에 의한 중력이상의 계산식이 유도되고, 직각 prism과 원기둥 모양의 물체에 대하여 중력을 계산하며 정확한 계산과 비교 연구를 한다. 근거리에 있는 source의 multipole확장은 표면적분보다 체적적분이 더 좋은 근사치를 얻게한다. 그렇지만 지구물리탐사의 실제에 있어 흔히 나타나는 원거리의 source에 대해서는 이 두 접근 방법은 서로 일치하게 됨을 알 수 있다.

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