QUOTIENT ADDITIVE PROPERTIES IN TOPOLOGICAL SPACES

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1. Introduction

Suppose that \( R \) is an equivalence relation on a space \( X \) and \([x]\) and \( X/R \) are compact for each \( x \) in \( X \). If the projection map \( \rho : X \to X/R \) is closed, then \( X \) is compact. See theorem 4.2.

This result is not new. This paper is concerned with the more general question: If \( R \) is an equivalence relation on a space \( X \) and if \( P \) is a topological property, does \( X \) have property \( P \) when \( X/R \) and \([x]\) have property \( P \) for each \( x \) in \( X \)?

We will call \( P \) a quotient additive property when the answer is in the affirmative.

The following properties are shown to be quotient additive: indiscreteness, discreteness, \( T_0 \), \( T_1 \), connectedness, total disconnectedness and singleton path components.

The following properties fail to be quotient additive even when \( \rho : X \to X/R \) is an open map: Lindelof, countable compactness, extremally disconnected and cofinite.

The following properties fail to be quotient additive even when \( \rho : X \to X/R \) is both open and closed: paracompact, metacompact, Hausdorff, regular and normal.

The following properties fail to be quotient additive even when \( \rho : X \to X/R \) is closed: second axiom, separable, metrizable and path connected.

If \( \rho : X \to X/R \) is closed, the following properties are shown to be quotient additive: Lindelof, countably compactness and (if \( X \) is first axiom) sequential compactness.

In section 5, \( X \times Y \) is shown to be a fixed point space if \( X \) is a \( T_1 \) fixed point space and \( Y \) is strongly separable (see definition 5.1).

2. General properties

We begin with
Theorem 2.1. If $X/R$ and $[x]$ are indiscrete for each $x$ in $X$, then $X$ is indiscrete.

Proof. Suppose that $O \neq O = X$ and $O$ is open in $X$. Then for each $x$ in $X$, $[x] \subseteq O$ or $[x] \cap O = \emptyset$ since $[x]$ is indiscrete. Let $O^* = \{[x] : [x] \subseteq O\}$. Then $O = O^* = X/R$ and $O^*$ is open in $X/R$, a contradiction.

Theorem 2.2. Let $X/R$ be discrete and let $[x]$ be discrete for each $x$ in $X$. Then $X$ is discrete.

Proof. $\{x\}$ is open in $[x]$ and $[x]$ is open in $X$ since $[[x]]$ is open in $X/R$. Thus $\{x\}$ is open in $X$.

Theorem 2.3. Let $X/R$ be a $T_0$-space and let $[x]$ be a $T_0$-space for each $x$ in $X$. Then $X$ is a $T_0$-space.

Proof. Let $x \neq y$. Case 1. $[x] \neq [y]$. Then we can assume that there exists an open set $O^*$ in $X/R$ such that $[x] \in O^*$ and $[y] \notin O^*$. Then $x \in p^{-1}[O^*]$ and $y \notin p^{-1}[O^*]$. Case 2. $[x] = [y]$. Then there exists an open set $O$ in $X$ such that $x \in O \cap [x]$ and $y \notin O \cap [x]$. Then $x \in O$ and $y \notin O$.

Theorem 2.4. Let $X/R$ and $[x]$ be $T_1$-spaces for each $x$ in $X$. Then $X$ is a $T_1$-space.

Proof. Modify theorem 2.2.

We note that neither $T_2$ nor $T_3$ nor $T_4$ nor $T_5$ may be substituted for $T_1$ in theorem 2.4 (see example 3.9).

Theorem 2.5. Let $X/R$ and $[x]$ be connected for each $x$ in $X$. Then $X$ is connected.

Proof. Suppose that $X = O \cup V$ where $O$ and $V$ are nonempty disjoint sets in $X$. Then $x \in O$ implies that $[x] \subseteq O$ and $y \in V$ implies that $[y] \subseteq V$. Let $O^* = \{[x] : x \in O\}$ and $V^* = \{[y] : y \in V\}$. Then $X/R = O^* \cup V^*$ and $O^*$ and $V^*$ are disjoint nonempty open sets in $X/R$, a contradiction.

Theorem 2.6. Let $X/R$ and $[x]$ be totally disconnected for each $x$ in $X$. Then $X$ is totally disconnected.

Proof. Let $A \subseteq X$ and suppose that $A$ has more than one point. Case 1. $A \subseteq [x]$ for some $x$ in $X$. Then $A$ is disconnected since $[x]$ is totally disconnected. Case 2. $A \subseteq [x]$ for no $x$ in $X$. Then $p[A]$ contains more than
one point in $X/R$ and hence is disconnected. It follows then that $A$ is disconnected.

**THEOREM 2.7.** Let $X/R$ and $[x]$ have singleton path components for each $x$ in $X$. Then $X$ has singleton path components.

**PROOF.** Let $f : [0, 1] \rightarrow X$ be continuous. We will show that $f$ is a constant. Now $p \circ f : [0, 1] \rightarrow X/R$ is a constant and hence $f([0, 1]) \subseteq [x]$ for some $x$ in $X$. Since $[x]$ has singleton path components, $f$ is a constant.

3. Some examples

The next two lemmas will be useful for counterexample purposes.

**LEMMA 3.1.** Let $W = X \times Y$ and let $W/R = \{A_x : x \in X\}$ where $A_x = \{x\} \times Y$. Then $W/R$ is homeomorphic to $X$ and $A_x$ is homeomorphic to $Y$ for each $x$ in $X$.

**PROOF.** Let $p : W \rightarrow X$ via $p((x, y)) = x$ and let $q : W \rightarrow W/R$ via $q((x, y)) = A_x$. Let $h : W/R \rightarrow X$ via $h(A_x) = x$. Then $h$ is clearly bijective and $h \circ q = p$. Since $q$ and $p$ are quotient maps, $h$ is a homeomorphism.

That $A_x$ is homeomorphic to $Y$ is well known.

**LEMMA 3.2.** Let $X = I \times I$ where $I = [0, 1]$ and let $X$ have the dictionary order topology. Let $X/R = \{A_x : x \in I\}$ where $A_x = \{x\} \times I$. Then $A_x$ is homeomorphic to $I$ for each $x$ in $X$ and $X/R$ is homeomorphic to $I$.

**PROOF.** Let $p : X \rightarrow I$ by $p(x, y) = x$. $p$ is clearly continuous and onto, and since $X$ is compact and $I$ is hausdorff, $p : X \rightarrow I$ is a closed map and hence a quotient map. Let $q : X \rightarrow X/R$ by $q(x, y) = A_x$. Since $X/R$ is given the quotient topology, $q : X \rightarrow X/R$ is a quotient map. Let $h : X/R \rightarrow I$ by $h(A_x) = x$. As in lemma 3.1, $h$ is a homeomorphism.

**DEFINITION 3.3.** A property $p$ will be termed *finitely nonproductive* if the product of two spaces with property $p$ need not have property $p$.

**THEOREM 3.4.** Let $p$ be a finitely nonproductive property. If $X/R$ and $[x]$ have property $p$ for each $x$ in $X$, then $X$ need not have property $p$ even if $p : X \rightarrow X/R$ is open.

**PROOF.** Use lemma 3.1.

**COROLLARY 3.5.** Let $p$ be any one of the following properties: Lindelof, count-
ably compact, paracompact, extremally disconnected, normal or cofinite. If $X/R$ and $[x]$ have property $p$ for each $x$ in $X$, then $X$ need not have property $p$ even if $p : X \to X/R$ is open.

**PROOF.** All of the above properties are finitely nonproductive.

**COROLLARY 3.6.** Let $X, X/R$ and $[x]$ be as in lemma 3.2. Then $X/R$ and $[x]$ are separable, second axiom, metrizable and path connected. $X$ has none of these properties and $p : X \to X/R$ is closed.

**EXAMPLE 3.7.** Let $X=\{x : 0 \leq x < 1$ or $1 < x < 2$ and $x$ is rational$\}$. Let $Y=\{0, 1\}$ and let $f : X \to Y$ as follows: $f(x)=x$ if $0 \leq x < 1$ $f(x)=x-1$ if $1 < x < 2$. Let $X$ have the usual topology and let $Y$ have the quotient topology. For each $y$ in $Y$, $f^{-1}[y]$ is a singleton set or a doubleton set and hence is compact, locally compact, locally connected and sequentially compact. $Y$ also has all of these properties. $X$ has none of these properties.

**EXAMPLE 3.8.** Let $X=\{a, 1, 2, \ldots, n, \ldots \}$ and let a subset of $X$ be open iff it is empty or contains $a$. Let $R$ be the equivalence relation whose equivalence classes are $[a]=[a]$ and $[1]=[1,2,\ldots,n,\ldots]$. Now $[a]$ and $[1]$ are paracompact and metacompact. $X/R$ is a two point space and hence is metacompact and every open cover has a locally finite open refinement. But $\{[a, x] : x \in X\}$ is an open cover of $X$ with no refinement which is locally finite or point finite. Note that $p : X/R$ is both open and closed.

**EXAMPLE 3.9.** Let $X=\{a, b, 1, 2, 3, \ldots, n, \ldots \}$ and let a subset $A$ of $X$ be open iff $A \cap [a, b]=\emptyset$ or $A \cap [a, b]=\emptyset$ and $\mathcal{C}A$ is finite, $\mathcal{C}$ denoting the complement operator. Let $[a]=[a, b]$ and $[n]=[n]$ for $n=1,2,\ldots$. Then $X/R$ is homeomorphic to $\{0, 1, 1/2, \ldots, 1/n, \ldots \}$ with the usual topology and $p : X \to X/R$ is both open and closed and $X/R$ and $[x]$ are $T_2$, $T_3$, $T_4$ and $T_5$. $X$ has none of these properties.

4. $p : X \to X/R$ closed

When $p : X \to X/R$ is a closed map, certain covering properties are well behaved.

We begin with a well known property of closed maps.

**LEMMA 4.1.** Let $f : X \to Y$ be a closed map and let $f^{-1}[y] \subseteq O$, $O$ being an open set in $X$. Then there exists an open set $U$ in $Y$ such that $y \in U$ and $f^{-1}[U] \subseteq O$. 
PROOF. Let \( U = E^f[O] \).

THEOREM 4.2. Let \( p : X \to X/R \) be closed and let \( X/R \) and \([x]\) be compact for each \( x \) in \( X \). Then \( X \) is compact.

PROOF. See [2], theorem 5.3, page 236 or modify the proof of the next theorem.

THEOREM 4.3. If \( p : X \to X/R \) is a closed map and if \( X/R \) and \([x]\) are Lindelöf for each \( x \) in \( X \), then \( X \) is Lindelöf.

PROOF. Let \( X = \{O_\alpha : \alpha \in \Lambda \} \) where \( O_\alpha \) is open in \( X \). Since \([x]\) is a Lindelöf space, there exists a countable set \( \Lambda([x]) \subseteq \Lambda \) such that \( p^{-1}([x]) = [x] \subseteq \bigcup \{O_\alpha : \alpha \in \Lambda([x])\} \). By lemma 4.2, there exists an open set \( U([x]) \) in \( X/R \) such that \( [x] \subseteq U([x]) \) and \( p^{-1}[U([x])] \subseteq \bigcup \{O_\alpha : \alpha \in \Lambda([x])\} \). Since \( X/R \) is Lindelöf, \( X/R = \bigcup \{U([x]) : i \geq 1\} \) and thus \( X = \bigcup \{O_\alpha : \alpha \in \bigcup \{\Lambda([x]) \} : i \geq 1\} \).

THEOREM 4.4. If \( p : X \to X/R \) is a closed map and if \( X/R \) and \([x]\) are countably compact for each \( x \) in \( X \), then \( X \) is countably compact.

PROOF. Let \( \{E_n : n \geq 1\} \) be a sequence of closed sets with the finite intersection property. We may assume that \( E_n \supseteq E_{n+1} \) for all \( n \). Now \( \{p[E_n] : n \geq 1\} \) is a sequence of closed sets in \( X/R \) with the finite intersection property. Since \( X/R \) is countably compact, there exists a \([x]\) in \( X/R \) such that \([x] \subseteq \bigcap \{p[E_n] : n \geq 1\} \). Then \( \{[x] \cap E_n : n \geq 1\} \) is a sequence of sets closed in \([x]\) with the finite intersection property and since \([x]\) is countably compact, \( \bigcap \{[x] \cap E_n : n \geq 1\} \neq \emptyset \). Thus \( \bigcap \{E_n : n \geq 1\} \neq \emptyset \) and \( X \) is countably compact.

COROLLARY 4.5. If \( p : X \to X/R \) is closed and if \( X/R \) and \([x]\) are sequentially compact for each \( x \) in \( X \) and if \( X \) is a first axiom space, then \( X \) is a first axiom space.

PROOF. \( X/R \) and \([x]\) are countably compact for each \( x \) in \( X \). Then \( X \) is countably compact by theorem 4.4 and hence \( X \) is sequentially compact since first axiom is assumed.

5. A fixed point theorem

We first note that \( X \times Y \) need not be a fixed point space when \( X \) and \( Y \) are fixed point spaces. See [1].

Using some of the earlier ideas, we will give a sufficient condition for
X \times Y to be a fixed point space.

DEFINITION 5.1. A space $X$ will be called strongly separable iff there exists a point $x^*$ in $X$ such that $X = c(\{x^*\})$ and $x \neq x^*$ implies that $\{x\}$ is closed. $c$ denotes the closure operator.

EXAMPLE 5.2. Let $X$ be a set and $x^* \in X$. Let $\mathcal{F}_1 = \{0 \subseteq X: 0 = O \text{ or } x^* \not\in 0\}$. Let $\mathcal{F}_2 = \{0 \subseteq X: 0 = O \text{ or } x^* \not\in 0 \text{ and } \emptyset \neq 0 \text{ is finite}\}$. Let $\mathcal{F}_3$ be any topology on $X$ for which $\mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_1$. Then $(X, \mathcal{F}_3)$ is strongly separable.

LEMMA 5.3. Let $X$ be a strongly separable space with distinguished point $x^*$. Then $X$ is a fixed point space.

PROOF. Let $f: X \to X$ be continuous. If $f(x^*) = x^*$, we are done. So assume that $f(x^*) = y \neq x^*$. Then $f(y) \in f(c(\{x^*\})) \subseteq c(f(\{x^*\})) = c(\{y\}) = \{y\}$. Thus $f(y) = y$.

LEMMA 5.4. Let $X$ be a space and $\{A_\alpha : \alpha \in \Delta\}$ a partition of $X$. Suppose further that for each $\alpha \in \Delta$, there is a point $x_\alpha \in X$ such that $A_\alpha = c(\{x_\alpha\})$. If $f: X \to Y$ is continuous and $\alpha \in \Delta$, there exists a $\beta \in \Delta$ such that $f[A_\alpha] \subseteq A_\beta$.

PROOF. Let $f(x_\alpha) \in A_\beta$. Then $f[A_\alpha] = f[c(\{x_\alpha\})] \subseteq c(f[\{x_\alpha\}]) \subseteq A_\beta = A_\beta$.

THEOREM 5.5. Let $X$ be a $T_1$-fixed point space and let $Y$ be a strongly separable space with distinguished point $y^*$. Then $X \times Y$ is a fixed point space.

PROOF. Let $R$ be the equivalence relation induced by the partition $\{(x) \times Y : x \in X\}$. Note that $(x) \times Y = c((x, y^*))$. Now let $f: X \times Y \to X \times Y$ be continuous and let $f/R: (X \times Y)/R \to (X \times Y)/R$ be the map induced by lemma 5.4. $f/R$ is continuous. By lemma 3.1, $(X \times Y)/R$ is homeomorphic to $X$ and hence is a fixed point space. There exists then an $x$ in $X$ such that $f/R([x] \times Y) = [x] \times Y$. Then $f([x] \times Y: [x] \times Y \to [x] \times Y$ and since $[x] \times Y$ is a fixed point space, $f(x, y) = (x, y)$ for some $y$ in $Y$.

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REFERENCES
