ON SEMI-WEAKLY CONTINUOUS MAPPINGS

By T. Noiri and B. Ahmad

1. Introduction

In 1961, N. Levine [1] introduced the concept of weakly continuous mappings. P.E. Long and D.A. Carnahan [3] studied several properties of almost continuous mappings in the sense of Singal [6]. T. Noiri [4] pointed out that the word "almost continuous" can be replaced by "weakly continuous" in some theorems of [3]. The purpose of this note is to introduce a new class of mappings called semi-weakly continuous mappings and investigate some properties analogous to those given in [4] concerning weakly continuous mappings.

2. Preliminaries

Let X be a topological space and S be a subset of X. A subset S is said to be semi-open [2] if there exists an open set U such that $U \subseteq S \subseteq \text{Cl}(U)$, where Cl(U) denotes the closure of U. The complement of a semi-open set is called semi-closed. The union of all semi-open sets of X contained in S is called the semi-interior of S and denoted by $s\text{Int}(S)$. The intersection of all semi-closed sets of X containing S is called the semi-closure of S and denoted by $s\text{Cl}(S)$. A mapping $f : X \rightarrow Y$ is said to be weakly continuous [1] (resp. almost continuous [6]) if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subseteq \text{Cl}(V)$ (resp. $f(U) \subseteq \text{Int}(\text{Cl}(V))$), where Int(S) is the interior of S.

3. Semi-weakly continuous mappings

DEFINITION 1. A mapping $f : X \rightarrow Y$ is called semi-weakly continuous (briefly s.w.c.) if for each $x \in X$ and each open set V containing $f(x)$ there exists a semi-open set U containing x such that $f(U) \subseteq s\text{Cl}(V)$.

A mapping $f : X \rightarrow Y$ is said to be semi-continuous [2] if for each open set V of Y, $f^{-1}(V)$ is semi-open in X. In [2, Theorem 12], it is known that a mapping $f : X \rightarrow Y$ is semi-continuous if and only if for each $x \in X$ and each open set V containing $f(x)$ there exists a semi-open set U containing x such that $f(U) \subseteq V$. Therefore, every semi-continuous mapping is s.w.c., but the converse is not
true as the following example shows.

EXAMPLE 1. Let $X$ and $Y$ be both the set of real numbers. Let $\tau$ be the usual topology for $X$ and $\sigma$ the cocountable topology for $Y$. Then the identity mapping $f: (X, \tau) \to (Y, \sigma)$ is semi-weakly continuous and not semi-continuous.

THEOREM 1. A mapping $f: X \to Y$ is s.w.c. if and only if for every open set $V$ in $Y$ $f^{-1}(V) \subset s\text{Int}(f^{-1}(s\text{Cl}(V)))$.

PROOF. Let $x \in X$ and $V$ an open set containing $f(x)$. Then $x \in f^{-1}(V) \subset s\text{Int}(f^{-1}(s\text{Cl}(V)))$. Put $U = s\text{Int}(f^{-1}(s\text{Cl}(V)))$. Then $U$ is semi-open and $f(U) \subset s\text{Cl}(V)$. Conversely, let $V$ be any open set of $Y$ and $x \in f^{-1}(V)$. Then there exists a semi-open set $U$ in $X$ such that $x \in U$ and $f(U) \subset s\text{Cl}(V)$. Therefore, we have $x \in U \subset f^{-1}(s\text{Cl}(V))$ and hence $x \in s\text{Int}(f^{-1}(s\text{Cl}(V)))$. This proves that $f^{-1}(V) \subset s\text{Int}(f^{-1}(s\text{Cl}(V)))$.

THEOREM 2. Let $f: X \to Y$ be a mapping and $g: X \to X \times Y$ be the graph mapping of $f$, given by $g(x) = (x, f(x))$ for every point $x \in X$. If $g$ is s.w.c., then $f$ is s.w.c.

PROOF. Let $x \in X$ and $V$ any open set containing $f(x)$. Then $X \times V$ is an open set in $X \times Y$ containing $g(x)$. Since $g$ is s.w.c., there exists a semi-open set $U$ containing $x$ such that $g(U) \subset s\text{Cl}(X \times V)$. It follows from Lemma 4 of [5] that $s\text{Cl}(X \times V) \subset X \times s\text{Cl}(V)$. Since $g$ is the graph mapping of $f$, we have $f(U) \subset s\text{Cl}(V)$. This shows that $f$ is s.w.c.

THEOREM 3. If $f: X \to Y$ is a s.w.c. mapping and $Y$ is Hausdorff, then the graph $G(f)$ is a semi-closed set of $X \times Y$.

PROOF. Let $(x, y) \in G(f)$. Then, we have $y \neq f(x)$. Since $Y$ is Hausdorff, there exist disjoint open sets $W$ and $V$ such that $f(x) \in W$ and $y \in V$. Since $f$ is s.w.c., there exists a semi-open set $U$ containing $x$ such that $f(U) \subset s\text{Cl}(W)$. Since $W$ and $V$ are disjoint, we have $V \cap s\text{Cl}(W) = \emptyset$ and hence $V \cap f(U) = \emptyset$. This shows that $(U \times V) \cap G(f) = \emptyset$. It follows from Theorems 2 and 11 in [2] that $G(f)$ is semi-closed.

DEFINITION 2. By a s.w.c. retraction, we mean a s.w.c. mapping $f: X \to A$, where $A \subset X$ and $f|A$ is the identity mapping on $A$.

THEOREM 4. Let $A \subset X$ and $f: X \to Y$ be a s.w.c. retraction of $X$ onto $A$. If $X$ is a Hausdorff space, then $A$ is a semi-closed set in $X$. 
PROOF. Suppose that $A$ is not semi-closed. Then there exists a point $x \in s\text{Cl} (A) - A$. Since $f$ is s.w.c. retraction, we have $f(x) \neq x$. Since $X$ is Hausdorff, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $f(x) \in V$. Thus we get $U \cap s\text{Cl} (V) = \emptyset$. Now, let $W$ be any semi-open set in $X$ containing $x$. Then $U \cap W$ is a semi-open set containing $x$ and hence $(U \cap W) \cap A \neq \emptyset$ because $x \in s\text{Cl} (A)$. Let $y \in (U \cap W) \cap A$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \in s\text{Cl} (V)$. This gives that $f(W) \subseteq s\text{Cl} (V)$. This contradicts that $f$ is s.w.c. Hence $A$ is semi-closed in $X$.

4. S-connected space

DEFINITION 3. A space $X$ is said to be $S$-connected [7] if $X$ can not be written as the disjoint union of two non-empty semi-open sets.

Every $S$-connected space is connected but the converse is not true as the following example shows.

EXAMPLE 2. Let $X = \{a, b, c\}$ and $\tau = (\emptyset, X, \{a\}, \{b\}, \{a, b\})$. Then $(X, \tau)$ is a connected space. However, it is not $S$-connected.

It is shown in Theorem 4 of [3] (resp. Theorem 3 of [4]) that connectedness is invariant under almost continuous (resp. weakly continuous) surjections. It is also known that $S$-connectedness is invariant under semi-continuous surjections. However, we have the following.

THEOREM 5. If $X$ is an $S$-connected space and $f : X \to Y$ is a s.w.c. surjection, then $Y$ is connected.

PROOF. Suppose that $Y$ is not connected. Then there exist non-empty open sets $V_1$ and $V_2$ of $Y$ such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence, we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$, $f^{-1}(V_1) \cup f^{-1}(V_2) = X$ and $f^{-1}(V_i) \neq \emptyset$ because $f$ is surjective. By Theorem 1, we have

$$f^{-1}(V_i) \subseteq s\text{Int}(f^{-1}(s\text{Cl}(V_i))), \quad i = 1, 2.$$  

Since $V_i$ is open and closed, we obtain $f^{-1}(V_i) \subseteq s\text{Int}(f^{-1}(V_i))$ and hence $f^{-1}(V_i)$ is semi-open for $i = 1, 2$. This implies that $X$ is not $S$-connected. Therefore $Y$ is connected.

THEOREM 6. If $X$ is an $S$-connected space and $f : X \to Y$ is a semi-continuous mapping with the closed graph, then $f$ is constant.
PROOF. Suppose that $f$ is not constant. There exist distinct points $x_1, x_2$ in $X$ such that $f(x_1) \neq f(x_2)$. Since the graph $G(f)$ is closed and $(x_1, f(x_2)) \notin G(f)$, there exist open sets $U$ and $V$ containing $x_1$ and $f(x_2)$, respectively, such that $f(U) \cap V = \emptyset$. Since $f$ is semi-continuous, $U$ and $f^{-1}(V)$ are disjoint non-empty semi-open sets. It follows from Theorem 17 of [7] that $X$ is not $S$-connected. Therefore, $f$ is constant.

COROLLARY (Thompson [7]). Let $X$ be irreducible. If $f : X \to Y$ is a continuous mapping with the closed graph, then $f$ is constant.

PROOF. Since every continuous mapping is semi-continuous, this is an immediate consequence of Theorem 17 in [7] and Theorem 6.

Yatsushiro College of Technology  
Faculty of Science  
Yatsushiro, Kumamoto  
Garyounis University  
Benghazi, Libya

REFERENCES


