A CHARACTERIZATION OF GROUPS AND LEFT GROUPS

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A semigroup $S$ is said to be a left group if, for every couple $a, b \in S$ there exists a unique element $x$ of $S$ such that $xa = b$. It is well known that a semigroup $S$ is a left group if and only if $S$ is left simple and right cancellative; or, equivalently, $S$ is regular and the set $E(S)$ of idempotent elements of $S$ is a left zero semigroup. The present author [2] has recently proved that a semigroup $S$ is a left group if and only if $B(ab) = B(a)$ holds for every couple $a, b$ in $S$, where $B(a)$ is the principal bi-ideal of $S$ generated by the element $a$ of $S$. Another characterization by the author reads as follows.

LEMMA 1. A semigroup $S$ is a left group if and only if the multiplicative semigroup $B(S)$ of all bi-ideals of $S$ is a left zero band.

We need also the following criterion of this author [3].

LEMMA 2. A semigroup $S$ is a semilattice of left groups if and only if $B(S)$ is a left regular band.

First we shall prove the following result.

THEOREM 1. A semigroup $S$ is a left group if and only if the condition

$$ (1) \quad BAL = B $$

holds for every left ideal $L$ and for every couple $A, B$ in $B(S)$.

PROOF. The necessity follows at once from Lemma 1. Conversely, if $S$ is a semigroup having property (1) for every left ideal $L$ and for all $A, B \in B(S)$, then (1) implies $B = B^2S$ for every bi-ideal $B$ of $S$, whence every bi-ideal $B$ of $S$ is a right ideal of $S$. Next we show that $S$ is regular. (1) implies $R = R^2S$ for any right ideal $R$ of $S$. Hence it follows that $R \subseteq R^2$, and thus $R = R^2$. Therefore the bi-ideal semigroup $B(S)$ is a band, and $S$ is regular. But $S$ is a left duo regular semigroup, which is a semilattice of left groups. By our Lemma 2, for any left ideal $L$ of $S$, we have $SLS = SL = L$. Hence $L = S$, by

* The Hungarian word BAL means LEFT in English.
(1), and $S$ is left simple. Therefore (1) implies $R_1R_2S=R_1R_2=R_1$ for every couple $R_1, R_2 \in B(S)$. Finally, Lemma 1 implies that $S$ is a left group.

**THEOREM 2.** A semigroup $S$ is a group if and only if the equality

$$B = ABL$$

holds for every left ideal $L$ and for all $A, B \in B(S)$.

**PROOF.** Necessity is trivial because of a group has no proper bi-ideal. Sufficiency: (2) implies $B = SBS$ for any bi-ideal $B$ of $S$. Hence every bi-ideal $B$ is a two-sided ideal of $S$. Now (2) implies $I = I^3 = I^2$ for every two-sided ideal $I$ of $S$. Hence it follows that every two-sided ideal of $S$ is globally idempotent and thus $S$ is a regular duo semigroup. This means (cf. [4]), that $S$ is a semilattice of groups. Then, for any ideal $I$ of $S$,

$$S = IS = IS = I.$$

Therefore $S$ is the only bi-ideal of $S$, whence it follows that $S$ is a group, indeed.

Finally we formulate the left-right dual of Theorem 1.

**THEOREM 3.** A semigroup $S$ is a right group if and only if the condition $B = RAB$ holds for all $A, B \in B(S)$ and for every right ideal $R$ of $S$.

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**REFERENCES**