THE $T_1$-CONTINUOUS FUNDAMENTAL GROUP OF A CERTAIN FINITE SPACE

By Karl R. Gentry and Hughes B. Hoyle, III

1. Introduction

Let $X$ be a topological space and let $x_0 \in X$. Then $C(X, x_0)$ will be used to denote the set of all continuous loops in $X$ at $x_0$. The idea of using continuous functions as relating functions on $C(X, x_0)$ to get an equivalence relation on $C(X, x_0)$ has long been in existence, and extensive studies have been made of the resulting homotopy groups. In [5], we considered using certain types of non-continuous functions as relating functions on $C(X, x_0)$. In particular an admitting homotopy relation $N$ was defined, which in general, turned out to be a larger class of relating functions than the class of continuous functions. Most types of non-continuous functions, including almost continuous functions [1], $C$-continuous functions [2], connectivity maps [6], and $T_1$-continuous functions [4], provide an admitting homotopy relation. Also in [5], it was shown how an admitting homotopy relation $N$ could be used to obtain a generalized homotopy group $N(X, x_0)$. The question has been raised as to an example of when one of these generalized homotopy groups is different from the corresponding usual homotopy group. In this paper we let $N$ be the admitting homotopy relation $T_1$-continuous and give an example of a space $X$ and a point $x_0 \in X$ such that the $T_1$-continuous fundamental group $N(X, x_0)$ is different from the fundamental group $\pi_1(X, x_0)$. That is if the relating functions between the loops are only required to be $T_1$-continuous, then we get a different group than if we required the relating functions between the loops to be continuous.

Throughout this paper I will be used to denote the closed unit interval with the usual topology.

2. The example

EXAMPLE. Let $X = \{a, b, c, d\}$, and let $T = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Then $\pi_1(X, b)$ is not isomorphic to $N(X, b)$. 

PROOF. Let \( f : I \rightarrow X \) be the continuous function defined by \( f(x) = b \) for all \( x \in I \) and let \( g : I \rightarrow X \) be a continuous function such that \( g(0) = b = g(1) \). Then since \( g \) is continuous and \( \{a, b, c\} \in T \), \( g^{-1}(\{a, b, c\}) \) is open in \( I \) and thus \( D = \{x | g(x) = d\} \) is closed in \( I \). Similarly, \( A = \{x | g(x) = a\} \) is closed in \( I \).

Define \( F : I \times I \rightarrow X \) by

\[
F(x, t) = \begin{cases} 
  d & \text{if } x \in D \text{ and } 0 \leq t \leq 1/2 \\
  a & \text{if } x \in A \text{ and } 0 \leq t \leq 1/2 \\
  g(x) & \text{if } t = 0 \\
  b & \text{otherwise}
\end{cases}
\]

Then \( F \) is well-defined and clearly \( F(0, t) = b = F(1, t) \) for all \( t \in I \) and \( F(x, 0) = g(x) \) and \( F(x, 1) = f(x) \) for all \( x \in I \). We wish to show that \( F \) is \( T_1 \)-continuous.

Let \( \mathcal{U} \) be an open cover of \( X \). Then either \( X \in \mathcal{U} \) or \( \{a, b, c\} \) and \( \{b, c, d\} \) are in \( \mathcal{U} \). If \( X \in \mathcal{U} \), then an open cover of \( I \times I \) which will work is \( \{I \times I\} \). If \( \{a, b, c\} \) and \( \{b, c, d\} \) are in \( \mathcal{U} \), then an open cover of \( I \times I \) which will work is \( \{I \times I - D \times [0, 1/2], I \times I - A \times [0, 1/2]\} \). Hence, \( F \) is \( T_1 \)-continuous. It follows that \( N(X, b) \) is the trivial group.

We will now show that \( II_1(X, b) \) has at least two elements. Once again let \( f \) be the constant loop at \( b \) and define \( h : I \rightarrow X \) by

\[
h(x) = \begin{cases} 
  b & \text{if } 0 \leq x < 1/5 \\
  a & \text{if } 1/5 \leq x \leq 2/5 \\
  c & \text{if } 2/5 < x < 3/5 \\
  d & \text{if } 3/5 \leq x \leq 4/5 \\
  b & \text{if } 4/5 \leq x \leq 1
\end{cases}
\]

Now \( f \) and \( h \) are loops at \( b \) and we wish to show that \( f \) and \( h \) are not homotopic modulo \( b \). To this end suppose that there is a continuous function \( F : I \times I \rightarrow X \) such that \( F(x, 0) = h(x) \), \( F(x, 1) = f(x) \), and \( F(0, t) = b = F(1, t) \) for all \( x \in I \), \( t \in I \). Let \( p \) and \( q \) be the points \( p = (2/5, 0) \), \( q = (3/5, 0) \). Then \( J = (2/5, 3/5) \times (0) \). Since \( \{c\} \in T \), \( F^{-1}(\{c\}) \) is an open subset of \( I \times I \). Since \( F(x, 0) = h(x) \) for all \( x \in I \), \( F^{-1}(\{c\}) \) contains \( J \). Let \( U \) be the component of \( F^{-1}(\{c\}) \) which contains \( J \). Then \( U \) is open and connected and since \( F \) is \( h \) on \( I \times \{0\} \), \( f \) on \( I \times \{1\} \), and \( b \) on \( \{0\} \times I \) and \( \{1\} \times I \), the only points on the boundary of \( I \times I \) which are in \( U \) are in \( J \). Let \( B \) be the boundary of \( U \). Let \( W = I \times I - U \) and let \( M = W \cup B \cup J \). Then \( W \cup B \) is closed in \( I \times I \) and since \( p, q \in B, W \cup B \cup J \) is closed. Hence, \( M \) is closed. Since \( J \) is the intersection of the boundary of \( I \times I \) and \( U \), the boundary of \( I \times I \) is contained in \( M \). Let \( Q \) be the component of \( M \) which contains the boundary of \( I \times I \). Then \( Q \) is closed and connected. Since \( Q \) is
bounded, $Q$ is compact and hence a continuum. Since $I \times I$ is closed in the plane, $Q$ is a continuum in the plane. Since $J$ is a subset of the boundary of $I \times I$ and $U$ is an open, connected subset of $I \times I$ containing $J$, $U-J$ is connected. Now $U-J$ is a connected subset of the complement of $Q$. Let $\mathcal{C}$ be the component of the complement of $Q$ which contains $U-J$.

We wish to show that the boundary of $\mathcal{C}$ is a subset of $J$ union the boundary of $U$. Let $x \in \text{bd} \mathcal{C}$. Then $x \in M$ and thus $x \in W \cup B \cup J$. If $x \in B \cup J$, then clearly $x \in (\text{bd} U) \cup J$. Now suppose $x \in W$. Since $W$ is an open subset of $I \times I$, there is a disc $D$ in the plane such that $x \in D \cap (I \times I) \subset W$. Now $x \in \text{bd} \mathcal{C}$ and thus $x \in Q$. But since $D$ is connected and contains $x$ and $Q$ is the component containing $x$, $D \cap (I \times I) \subset Q$. Now $Q$ contains the boundary of $I \times I$ and $\mathcal{C}$ is a component of the complement of $Q$ which intersects the interior of $I \times I$. Hence, $\mathcal{C}$ is contained in the interior of $I \times I$ and thus $x$ is neither a point nor a limit point of $\mathcal{C}$. Therefore, $x \notin \text{bd} \mathcal{C}$. But this is impossible. Hence, $x \in W$. Thus, $\text{bd} \mathcal{C} \subset J \cup (\text{bd} U)$. By [12, Theorem 2.1, p. 105], since $\mathcal{C}$ is a bounded component of the complement of $Q$, the bd $\mathcal{C}$ is a continuum. Let $K$ be the boundary of $\mathcal{C}$. Let $L=K-J$. Then $L \subseteq \text{bd} U$ and we now wish to show that $L$ is connected. Since $p,q \in K$ and neither $p$ nor $q$ is in $J$, $p,q \in L$. Suppose $L$ is not connected. Then $L$ is the union of two non-empty, mutually separated sets $\mathcal{A}$ and $\mathcal{B}$ with $p$ in one of them. Say $p \in \mathcal{A}$. Suppose $q \in \mathcal{A}$. Then $K=(\mathcal{A} \cup J) \cup \mathcal{B}$. Now $\mathcal{A}$ and $\mathcal{B}$ are mutually separated. Since $\mathcal{C}$ is an open subset of $I \times I$ containing $J$ in its boundary, no point of $J$ is a limit point of $K-J$ and no point of $K-J$ is a limit point of $J$ except $p$ and $q$. But $p$ and $q$ are in $\mathcal{A}$. Hence, $J$ and $\mathcal{B}$ are mutually separated. Thus, $\mathcal{A} \cup J$ and $\mathcal{B}$ are non-empty, mutually separated sets. But this is impossible, since $K$ is connected. Thus, $q \in \mathcal{B}$. Now suppose $\mathcal{A}$ is not connected. Then $\mathcal{A}=\alpha \cup \beta$ where $\alpha$ and $\beta$ are non-empty mutually separated sets with $p \in \alpha$. Then $K=\beta \cup (\alpha \cup J \cup \mathcal{B})$ where these two sets once again are mutually separated. Thus, $\mathcal{A}$ is connected. Since $J$ is an open subset of $K$, $K-J$ is closed and thus $L$ is closed. Since $\mathcal{A}$ is a component of $L$, $\mathcal{A}$ is closed. Hence, $\mathcal{A}$ is a continuum. Similarly, $\mathcal{B}$ is a continuum. By [12, Theorem 3.1, 108], there is a simple closed curve $\Gamma$ in the plane such that $\Gamma$ separates $p$ from $q$ and $\Gamma \cap (\mathcal{A} \cup \mathcal{B})=\emptyset$. Let $Z$ be the boundary of $I \times I$ minus $J \cup \{p,q\}$. Then $J \cup \{p,q\}$ is a connected set containing $p$ and $q$ and since $\Gamma$ separates $p$ from $q$, $\Gamma \cap J \neq \emptyset$. Let $w \in \Gamma \cap J$. Similarly $\Gamma \cap Z \neq \emptyset$. Let $z \in \Gamma \cap Z$. Since $z \in \Gamma \cap Z$, there is a point $k$ in the unbounded component of the complement of the boundary of $I \times I$ such that $k \in \Gamma$ and the arc from $k$ to $z$ in $\Gamma$ not containing.
$w$ contains no point of $J$. Since $J$ is in the boundary of $\mathcal{O}$ there is a point $m \in \mathcal{O}$ such that $m \in I'$ and the arc from $k$ to $m$ in $I'$ containing $z$ contains no point of $J$. Let $A$ be the arc in $I'$ from $k$ to $m$ containing $z$. Then $A \cap J = \phi$ and since $I' \cap (\mathcal{A} \cup \mathcal{B}) = \phi$, $A \cap K = \phi$. But then the component of the compliment of $K$ containing $\mathcal{O}$ is not a subset of the interior of $I \times I'$, which is impossible. Hence, $L$ is connected. Since $L = K - J$ and $K \subset (bd \ U) \cup J$, $L \subset bd \ U$. Hence a connected subset of the boundary of $U$ contains both $p$ and $q$.

Let $P$ be the component of the boundary $B$ of $U$ which contains $p$ and $q$. Then since $B$ is closed, $P$ is closed.

Now $U$ was the component of $F^{-1}(c)$ containing $J$. Thus, no point of $B$ is in $F^{-1}(c)$, for if $x \in B$ and $F(x) = c$, then since $F$ is continuous at $x$, there is a disc $E$ such that $x \in E \cap (I \times I)$ and $F(E \cap (I \times I)) = \{c\}$. But $E \cap U \neq \phi$, since $x$ is in the boundary of $U$. Hence, $U \cup E$ is connected and $U$ was not maximal since $E$ must also contain a point not in $U$ since $x$ is in the boundary of $U$. No point of $B$ is in $F^{-1}(b)$, for if $x \in B$ and $F(x) = b$, then since $F$ is continuous at $x$, there is a disc $G$ such that $x \in G \cap (I \times I)$ and $F(G \cap (I \times I)) = \{b\}$. But $G$ contains no point of $F^{-1}(c)$ and hence no point of $U$. Hence, $F(B) \subset \{a, d\}$. But $F(p) = a$ and $F(q) = d$. Hence, $F(B) = \{a, d\}$. Since $\{a, b, c\} \subset T$, $F^{-1}(d)$ is closed. Similarly $F^{-1}(a)$ is closed. Since $P$ is closed, $P \cap F^{-1}(a)$ and $P \cap F^{-1}(d)$ are closed. But $P \subset B$ containing $p$ and $q$. Thus $P = (P \cap F^{-1}(a)) \cup (P \cap F^{-1}(d))$ which is a contradiction since $P$ is connected and $P \cap F^{-1}(a)$ and $P \cap F^{-1}(d)$ are non-empty closed sets. Thus, no such continuous function $F$ can exist and $f$ and $h$ are not homotopic modulo $b$. Hence $\pi_1(X, b)$ has at least two elements and $N(X, b)$ cannot be isomorphic to $\pi_1(X, b)$.

The University of North Carolina
Greensboro, North Carolina 27412
U.S.A.

REFERENCES

The $T_\lambda$-continuous fundamental group of a certain space

Mathematicae, 76(1972), 9—17.


