ON THE INTEGRAL PROPERTIES OF CLASSICAL SOLUTIONS
OF A PARABOLIC EQUATION

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1. Introduction

Questions of classical and weak solutions and their regularity of linear second-order parabolic equations with different boundary conditions are extensively studied in [1], [2], [3], etc. It is well known that a situation can arise, in which the first mixed boundary problem for such an equation has a classical solution, but does not have a weak one. The classical solution is, at the same time, a weak one if it is subject to additional conditions of integral or differential character. We aim at investigating the conditions that must be imposed on the known functions of the problem, that is, the coefficients, right-hand side and the functions defining the initial and boundary conditions, in order that a classical solution of the problem has square integrable derivatives with respect to space and time variables. No conditions will be imposed on the lateral surface of the region. We note that similar ideas for the case of elliptic equations were considered in [4], [5] and [6].

In the Euclidean space $\mathbb{R}^{n+1}$ of points $(x, t)=(x_1, \ldots, x_n, t)$ we consider the cylindrical region $Q=G \times (0, T)$ of height $T>0$, where $G$ is a bounded or unbounded region in $\mathbb{R}^n$ with boundary $\partial G$. Let $S=\partial G \times (0, T)$ and $G_c=\{(x, t) \in Q : t=c\}$.

Throughout this paper, we shall employ usual notation for spaces of continuous and continuously differentiable functions and for Sobolev spaces [2]. $V^{1,0}(Q)$ denotes the Banach space of all functions in $W^{1,0}(Q)$, which are continuous with respect to $t$ in the $L^2(G)$-norm, that is, $\lim_{h \to 0} \|u(x, t+h)-u(x, t)\|_{L^2(G)}=0$, and such that

$$\|u\|_{V^{1,0}(Q)} = \max_{0 \leq t \leq T} \|u(x, t)\|_{L^2(G)} + \|Du\|_{L^2(Q)} < \infty,$$

where $D=(D_1, \ldots, D_n)$ and $D_i=\partial / \partial x_i, i=1, \ldots, n$, and $|Du|^2 = \sum_{i=1}^n |D_i u|^2$.

In $Q$ consider the differential operator $L$ defined by
\[ Lu = D_t u - D_t (a_{ij} D_j u) + a_i D_i u + au, \]

where \( D_t = \partial/\partial t \) and summation convention is understood. All coefficients of \( L \) are assumed to be continuous functions in \( Q \), and \( L \) is strictly parabolic in \( Q \), that is,

\[ v|\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \mu |\xi|^2, \quad v, \mu = \text{const} > 0, \]

for any real \( \xi \in \mathbb{R}^n \) and all \((x, t) \in Q\).

From here and on, \( u(x, t) \) will denote a classical solution from the space \( C^{2,1}(Q) \cap C(\overline{Q}) \) of the problem

\[
\begin{align*}
Lu &= f(x, t) \quad \text{in } Q, \\
\left. u \right|_{t=0} &= g(x), \quad \left. u \right|_{s=0} = 0,
\end{align*}
\]

where \( f \in C(\overline{Q}) \) and \( g \in C(\overline{G}) \). Clearly, \( u \in L^2(Q) \) in case of bounded \( Q \). If \( Q \) is unbounded, we have \( u \in L^2(G') \times (0, T) \) for any \( G' \subseteq G \).

2. Integrability of \( Du \) on interior cylinders of height \( T \)

In this section we prove the following auxiliary result.

**Lemma 1.** Let the operator \( L \) be strictly parabolic in \( Q \) and have coefficients \( a_{ij}, a \in L^\infty(Q) \), and let \( f \in L^2(Q) \). If problem (2) has a classical solution \( u \), then for any subregion \( G' \subseteq G \), we have \( u \in V^{1,0}(Q') \), where \( Q' = G' \times (0, T) \).

**Proof.** Let \( q(x) \in C^1(G) \) be a cut-off function with compact support in \( G \) and such that \( 0 \leq q \leq 1 \) and \( q = 1 \) on \( G' \). Multiply both sides of the equation \( Lu = f \) by \( q^2 u \) and integrate the result over the cylinder \( Q_{t_1, t_2} = G \times (t_1, t_2) \), where \( 0 < t_1 < t_2 \leq T \). Properties of \( q \) and \( u \) enable us to apply the formula of integration by parts to obtain

\[
\begin{align*}
\frac{1}{2} \int_{Q_{t_1, t_2}} q^2 u^2 dx &- \frac{1}{2} \int_{Q_{t_1, t_2}} q^2 u^2 dx + \int_{Q_{t_1, t_2}} [a_{ij} D_j u D_i (q^2 u) + a_i q^2 u D_i u] dt \\
&+ a_2 q^2 u^2 dt = \int_{Q_{t_1, t_2}} q^2 f u dx dt.
\end{align*}
\]

Using (1), Cauchy inequalities \( |ab| \leq \varepsilon a^2 + (1/\varepsilon) b^2 \) and \( |a_{ij} \alpha_i \beta_j| \leq (a_{ij} \alpha_i \alpha_j)^{1/2} (a_{ij} \beta_i \beta_j)^{1/2} \), and Schwarz inequality, we obtain

\[
\begin{align*}
\frac{1}{2} \int_{Q_{t_1, t_2}} q^2 u^2 dx &+ \frac{1}{2} \int_{Q_{t_1, t_2}} q^2 |Du|^2 dx dt \leq \frac{1}{2} \int_{Q_{t_1, t_2}} q^2 u^2 dx \\
&+ \frac{1}{2} \int_{Q_{t_1, t_2}} |Dq|^2 u^2 dx dt + \frac{1}{2} \int_{Q_{t_1, t_2}} |Dq|^2 u^2 dx dt + \frac{1}{2} \sup Q (a \\
&+ \frac{1}{4 \varepsilon^2} \sum_{i=1}^n a_i^2) \int_{Q_{t_1, t_2}} q^2 u^2 dx dt.
\end{align*}
\]
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Choosing $\varepsilon_1$ and $\varepsilon_2$ sufficiently small such that $\nu - \varepsilon_1 - \varepsilon_2 > 0$, and passing to the limit as $t_1 \to 0$ in this inequality, we get

$$
\int_{Q_{1/2}} q^2 |Du|^2 \, dx \, dt \leq C \left( \int_Q \left[ q^2 f^2 + q^2 u^2 + |Dq|^2 u^2 \right] \, dx \, dt + \int_G q^2 g^2 \, dx \right) < \infty,
$$

where $C$ depends only on $\nu$, $\mu$ and the coefficients of $L$, and is independent of $t_2$. Hence, $q|Du| \in L^2(Q)$, which together with the fact that $u \in C(\bar{Q})$, completes the proof of the lemma.

3. Main result

Let $G_m$, $m = 1, 2, \ldots$, be a sequence of subregions of $G$, for which $G_m \subset \subset G_{m+1}$, $\partial G_m$ are sufficiently smooth and $\lim_{m \to \infty} G_m = G$. Since $u \in C(\bar{Q})$ and $u = 0$ on $S$, we can, in the case of bounded $G$, choose these $G_m$ in such a way that

$$
|u| < \frac{1}{m} \quad \text{for all } (x, t) \in (G \setminus G_m) \times [0, T].
$$

(3)

The fulfillment of (3) in the case of unbounded $G$ may be achieved by additionally assuming that

$$
\lim_{|x| \to \infty} u(x, t) = 0 \quad \text{and} \quad u \in L^2(Q).
$$

(4)

In the cylinder $Q_m = G_m \times (0, T)$, whose lateral surface we denote by $S_m$, consider the problem

$$
\begin{align*}
Lu_m &= f \quad \text{in } Q_m, \\
u_m |_{t=0} &= h_m(x)g, \quad u_m |_{S_m} = 0,
\end{align*}
$$

(5)

where $h_m \in C^1(G)$, $0 \leq h_m \leq 1$, $h_m = 1$ on $G_m - 1$ and supp $h_m \subset \subset G_m$. If such $u_m$ exists, we denote by $u_m^*$ its extension to the whole $Q$, which is equal to zero in $Q \setminus Q_m$.

**LEMMA 2.** Let us assume in addition to the hypotheses of lemma 1, that $g \in \tilde{W}^{1,1}(G)$ and $\int_0^T \sup_{G} |D_i a_{ij}| \, dt < \infty$. Then $u_m^* \in \tilde{W}^{1,1}(Q)$ and there is a subsequence of $(u_m^*)^G$ weakly converging in the metric of $W^{1,0}(Q)$ to a weak solution $u^*$ in $\tilde{W}^{1,0}(Q)$ of problem (2). Moreover,

$$
|u^*|_{W^{1,0}(Q)} \leq C \left( ||f||_{L^1(Q)} + ||g||_{L^1(G)} \right),
$$

(6)

where $C$ is independent of $f$ and $g$.

**PROOF.** Properties of $h_m$ and $g$ imply $h_m g \in \tilde{W}^{1,1}(G_m)$. In view of this and our conditions on the coefficients and $f$, it follows from the well known existence
and recalling the properties of \( h_m \), we have from last inequality

\[
\int_{Q_m} (v D u_m + a_i D u_m D_i + v + a_i v D u_m + a u_m v) \, dx \, dt = \int_{Q_m} f \, dx \, dt
\]

for any \( v \in \dot{W}^{1,1}(Q_m) \). In this identity set \( v = u_m e^{-kt} \), where \( k \) is a positive constant to be determined later. Using (1), integration by parts and Cauchy inequality, we get

\[
\frac{\nu}{2} e^{-kT} \int_{Q_m} |Du_m|^2 \, dx \, dt + \left( \frac{k}{2} + \inf_{Q} a - \frac{1}{2} \right) \sum_{i=1}^{n} a_i \frac{\nu}{2} e^{-kT} u_m^2 \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_{Q_m} f^2 \, dx \, dt + \frac{1}{2} \int_{Q_m} h_m g^2 \, dx.
\]

Choosing \( k \) sufficiently large such that

\[
\frac{k}{2} + \inf_{Q} a - \frac{1}{2} \sup_{Q} \sum_{i=1}^{n} a_i > 0
\]

and recalling the properties of \( h_m \), we have from last inequality

\[
\int_{Q_m} (|Du_m|^2 + u_m^2) \, dx \, dt \leq C(||f||_{L^2(Q_m)} + ||g||_{L^2(G)}),
\]

where \( C \) is independent of \( m \), which shows that

\[
||u_m||_{\dot{W}^{1,1}(Q_m)} \leq C(||f||_{L^2(Q_m)} + ||g||_{L^2(G)}) \leq K,
\]

with \( K \) independent of \( m \). Hence, there is a subsequence of \( \{u_m^{*}\} \) weakly converging in the metric of \( \dot{W}^{1,1}(Q) \) to some function \( u^{*} \) in \( \dot{W}^{1,1}(Q) \). Properties of \( h_m \) and (3) imply

\[
|h_m g - g| = (1 - h_m) |g| < \frac{1}{m - 1} \text{ for all } x \in G.
\]

Thus, the traces \( h_m g \) of \( u_m^{*} \) on \( G_0 \) uniformly converge to \( g \) in \( G_0 \). Hence, \( u_m^{*} |_{t=0} = g \). By elementary considerations, we may show that \( u^{*} \) is a weak solution of problem (2) in the class \( \dot{W}^{1,0}(Q) \).

Now, we formulate and prove our main result.

**THEOREM.** Let all the hypotheses of lemma 2 be satisfied. If problem (2) has a classical solution \( u \), which in case of unbounded \( G \) satisfies the extra conditions (4), then \( u \in \dot{W}^{1,1}(Q) \) and, hence, coincides with the weak solution of problem (2) in \( \dot{W}^{1,1}(Q) \).

**PROOF.** In \( Q_m \) consider the problem

\[
\nu \frac{\partial u}{\partial t} - \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = f \quad \text{in } Q_m,
\]

where \( p \) and \( \nu \) are positive constants and \( f \) is a measurable function on \( Q_m \). Assume that \( \nu \neq 0 \) and \( \nu \neq 1 \).

Taking the pointwise view of parabolic equations, we obtain

\[
\nu \frac{\partial u}{\partial t} - \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = f \quad \text{in } Q_m,
\]

where \( p \) and \( \nu \) are positive constants and \( f \) is a measurable function on \( Q_m \). Assume that \( \nu \neq 0 \) and \( \nu \neq 1 \).

Hence, the traces \( h_m g \) of \( u_m^{*} \) on \( G_0 \) uniformly converge to \( g \) in \( G_0 \). Hence, \( u_m^{*} |_{t=0} = g \). By elementary considerations, we may show that \( u^{*} \) is a weak solution of problem (2) in the class \( \dot{W}^{1,0}(Q) \).

Thus, the traces \( h_m g \) of \( u_m^{*} \) on \( G_0 \) uniformly converge to \( g \) in \( G_0 \). Hence, \( u_m^{*} |_{t=0} = g \). By elementary considerations, we may show that \( u^{*} \) is a weak solution of problem (2) in the class \( \dot{W}^{1,0}(Q) \).
where \( p \) is some fixed number, for which \( p + \inf a > 0 \). By lemma 2, we have \( w_m \in \bar{W}^{1,1}_0(Q_m) \) and there is a subsequence of \( \{ w_m^* \} \) weakly converging in the metric of \( W^{1,0}_0(Q) \) to the weak solution \( u^* \) of problem (2). Since, under our assumptions, a solution in \( W^{1,0}_0(Q) \) is also a solution in \( W^{1,1}_0(Q) \), we have \( u^* \in \bar{W}^{1,1}_0(Q) \). Without loss of generality, we can assume that the sequence \( \{ w_m^* \} \) itself weakly converges in \( W^{1,0}(Q) \)-metric to \( u^* \).

Lemma 1 and properties of \( w_m \) imply that \( u - w_m \) is a weak solution in \( V^{1,0}(Q_m) \) of the problem

\[
\begin{align*}
L w_m + p w_m &= f + p u \quad \text{in } Q_m, \\
w_m |_{t=0} &= h_m g, \quad w_m |_{S_m} = 0,
\end{align*}
\]

Taking into consideration our conditions concerning the coefficients of \( L \) and the fact that \( p + a > 0 \) in \( Q \), we may apply the maximum principle for solutions of parabolic equations, as stated in theorem 7.2 in [2, Ch. III], to obtain in view of (3) that

\[
|u - w_m| \leq \sup_{S_m \cup G_m} |u - w_m| \leq \frac{1}{m-1}
\]

almost everywhere (a.e.) in \( Q_m \), where \( G_m = \{ (x, t) : x \in G_m, \ t = 0 \} \). This inequality, together with (3), shows that

\[
|u - w_m^*| \leq \frac{1}{m-1} \quad \text{a.e. in } Q.
\]

Hence, \( w_m^* \) uniformly converges to \( u \) a.e. in \( Q \). But we showed above that the same sequence weakly converges in \( W^{1,0}_0(Q) \)-metric to \( u^* \), so we must have \( u = u^* \) a.e. in \( Q \). Therefore, \( u \) is the weak solution in \( \bar{W}^{1,1}_0(Q) \) of problem (2) and the proof is complete.

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REFERENCES


Let $D$ be a bounded and star-shaped domain and $u(z, t)$ is continuous with $|t| \leq b$.

We denote $A^p$ or equivalent $A^p$ where $V$ at $z \in D$.

The Heisenberg group $H^p$ is defined where $V'$ at $t \in b$.

In Section 2, we prove the main result by using the following condition.