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A Study on Bayes Reliability Estimators of k out of m Stress-Strength Model.

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ABSTRACT

We study some Bayes esimators of the reliability of k out of m stress-strength model under quadratic loss and various prior distributions. We obtain Bayes estimators, Bayes risk, predictive bounds and asymtotic distribution of Bayes estimator. We investigate behaviours of Bayes estimator in moderate samples.

I. INTRODUCTION

Suppose a system consisting m components is successful in its mission if at least $k(1 \leq k \leq m)$ of these components survive a random stress. We assume that the component strengths $Y_1, \cdots, Y_m$ are independent with a common cumulative distribution function (cdf) $G \in F$ where F is the class of all continuous univariate cdf's. Further, the common stress X experienced by each component has cdf F $\in F$ and is assumed to be independent of the Y's. The system reliability is then given by

$$R_{k,m} = \text{Prob}(\text{at least } k \text{ of the } Y_1, \cdots, Y_m \text{ exceed } X)$$

$$= \sum_{j=k}^{m} \binom{m}{j} \int_{-\infty}^{\infty} G_{(x)}^{m-j} \left[1 - G(x)\right]^{j} \, dF(x)$$

Based on independent random samples from F and G, uniformly minimum variance unbiased estimator of the system reliability have been derived by Battacharyya and Johnson [1]. And Bayesian estimation of reliability for the special case $k = m = 1$ was considered earlier by Enis and Geisser [3] with the same assumption. The Present paper deals with the Bayesian estimation of $R_{k,m}$ based on the above assumption.

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In section 2, assuming quadratic loss, we drive a Bayes estimator of $R_{k,m}$ as well as an expression for its risk. We also drive an predictive bounds that at least k of the $Y_1, \cdots, Y_m$ exceed X, given pairs of observations, say (X_i, Y_j), $i = 1, \cdots, n_1$ and $j = 1, \cdots, n_2$.

In section 3, we drive the asymtotic distribution of Bayes estimator of $R_{k,m}$ and show its asymtotic equivalence with the maximum likelihood estimator (MLE) under the "vague prior".

In section 4, we investigate behaviour of the Bayes estimates for the moderate samples when the prior distributions are varing.

II. ESTIMATORS OF $R_{k,m}$

Suppose that $X_1, \cdots, X_{n_1}$ and $Y_1, \cdots, Y_{n_2}$ are independent random samples from F and G where F and G are exponential with scale parameters θ_1 and θ_1, respectively.
That is,

$$f_1(x|\theta_1) = \theta_1 e^{\theta_1 x}, \theta_1, x > 0$$

and

$$f_2(y|\theta_2) = \theta_2 e^{-\theta_2 y}, \theta_2, y > 0.$$

Bhattacharyya and Johnson [1] have derived an explicit form of system reliability

$$(2.1) \qquad R(\lambda) = R_{k,m} = 1 - \sum_{j=k}^{m} a_j \left(\frac{j}{\lambda + j}\right)$$

where $\lambda = \theta_1 / \theta_2$

and $$a_j = \frac{m!}{(k-1)!\,(j-k)!\,(m-j)!} \; \frac{1}{j} \; (-1)^{j-k}, \; j = k, \cdots, m.$$

We shall employ independent gammer conjugate prior distributions $G(\alpha_1, \beta_1)$, $G(\alpha_2, \beta_2)$ for θ_1 and θ_2, respectively, whose pdf's are given by

$$g_1(\theta_1) = \frac{\beta_1 \alpha_1}{\Gamma(\alpha_1)} \theta_1^{\alpha_1 - 1} e^{-\beta_1 \theta_1}, \quad \alpha_1, \beta_1 > 0,$$

$$g_2(\theta_2) = \frac{\beta_2 \alpha_2}{\Gamma(\alpha_2)} \theta_2^{\alpha_2 - 1} e^{-\beta_2 \theta_2}, \quad \alpha_2, \beta_2 > 0, \quad \text{and}$$

then the distribution of $\psi_j = \dfrac{j}{\lambda + j}$ in (2.1) is given by

$$(2.2) \qquad g(\psi_j) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (j\tau)^{-\alpha_2} \psi_j^{\alpha_2 - 1} (1 - \psi_j)^{\alpha_1 - 1} [1 - (1 - j\tau)^{-1}) \psi_j]^{-(\alpha_1 + \alpha_2)}$$

where $0 < \psi_j < 1$, $\tau = \beta_1 / \beta_2$.

The statistician may utilize his prior information concerning θ_1 and θ_2 by choosing values of α_1, α_2, β_1 and β_2. The statistician's prior belief concerning $R(\lambda)$, if it is manifested in his feelings about $E_p[R(\lambda)]$ (the prior expected value), is given by

$$R^0_{k,m} = E_p[R(\lambda)]$$

$$= 1 - \sum_{j=k}^{m} a_j E_p [\frac{j}{\lambda + j}]$$

$$= 1 - (\frac{\alpha_2}{\alpha_1 + \alpha_2}) [\sum_{j=k}^{r1} a_j (j\tau)^{\alpha 1} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_1, \alpha_1 + \alpha_2 + 1, 1 - j\tau)$$

(2.3)
$$+ \sum_{j=r2}^{m} a_j (j\tau)^{-\alpha 2} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_2 + 1, \alpha_1 + \alpha_2 + 1, 1 - (j\tau)^{-1})],$$

where $r_1 = \min(m, \tau^{-1})$, $r_2 = \max(k, \tau^{-1} + 1)$ and ${}_2F_1(a, b, c, x)$ is the hypergeometric function of the second kind (see [4]) which is defind by

$${}_2F_1(a, b, c, x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt.$$

$$c > b > 0, \; x < 1$$

Proposition 2.1. The prior variance of $R(\lambda)$ is given by

$$\sigma_0^2 = 1 + \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 + 1)} [\sum_{j=k}^{r1} a_j^2 (j\tau)^{\alpha 1} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_1, \alpha_1 + \alpha_2 + 2,$$

$$1 - j\tau) + \sum_{j=r2}^{m} a_j^2 (j\tau)^{-\alpha 2} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_2 + 2, \alpha_1 + \alpha_2 + 2, 1 - (j\tau)^{-1})]$$

$$+ 2 \frac{\alpha_2}{(\alpha_1 + \alpha_2)} [\sum_{j=k}^{r1} a_j b_j (j\tau)^{\alpha 1} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_1, \alpha_1 + \alpha_2 + 1, 1 - j\tau)$$

(2.4)
$$+ \sum_{j=r2}^{m} a_j b_j (j\tau)^{-\alpha 2} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_2 + 1, \alpha_1 + \alpha_2 + 1, 1 - (j\tau)^{-1})]$$

$$- [1 - (\frac{\alpha_2}{\alpha_1 + \alpha_2}) \{ \sum_{j=k}^{r1} a_j (j\tau)^{\alpha 1} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_1, \alpha_1 + \alpha_2 + 1, 1 - j\tau)$$

$$+ \sum_{j=r2}^{m} a_j (j\tau)^{-\alpha 2} {}_2F_1 (\alpha_1 + \alpha_2, \alpha_2 + 1, \alpha_1 + \alpha_2 + 1, 1 - (j\tau)^{-1} \}]^2$$

where $b_j = -1 + \sum_{i=k}^{m} \frac{i}{(1-j)} a_i$.

Proof. $R^2(\lambda) = [1 - \sum_{j=k}^{m} a_j (\frac{j}{\lambda + j})]^2$

$$= 1 - 2\sum_{j=k}^{m} a_j (\frac{j}{\lambda + j}) + \sum_{j=k}^{m} a_j^2 (\frac{j}{\lambda + j})^2 + \sum_{\substack{i=k \\ i \neq j}}^{m} \sum_{j=k}^{m} a_i a_j (\frac{1}{\lambda + j})(\frac{i}{\lambda + i})$$

Here,
$$\sum_{\substack{i=k \\ i \neq j}}^{m} \sum_{j=k}^{m} a_i a_j (\frac{j}{\lambda + j})(\frac{i}{\lambda + i})$$

$$= \sum_{\substack{i \\ i \neq j}} \sum_{j} a_i a_j \frac{ij}{i-j} (\frac{1}{\lambda + j} - \frac{1}{\lambda + i})$$

$$= \sum_{i \neq j} \sum a_i a_j \frac{i}{i-j} \left(\frac{j}{\lambda+j}\right) - \sum_{i \neq j} \sum a_i a_j \frac{i}{i-j} \left(\frac{i}{\lambda+i}\right)$$

$$E_p[R^2(\lambda)] = 1 + \sum_{j=k}^{m} a_j E_p\left(\frac{j}{\lambda+j}\right)^2 + 2 \sum_{j=k}^{m} a_j b_j E_p\left(\frac{j}{\lambda+j}\right) \cdot$$

From this, we can easily have the result.

In order to obtain a more useful indication of his prior belief, we note followings: $\psi_j = \dfrac{j}{\lambda+j}$ is a convex function of λ, then $R(\lambda)$ is a concave function of λ. Similary, let $w = \lambda^{-1}$ Then $\psi_j = \dfrac{j}{w^{-1}+j}$ is a concave function of w.

Thus, $R(\lambda) = W(w)$ is a convex function of w. Since

$$E_p(\lambda) = \frac{\alpha_1 \beta_2}{(\alpha_2 - 1)\beta_1} \quad \text{and} \quad E_p(w) = \frac{\alpha_2 \beta_1}{(\alpha_1 - 1)\beta_2},$$

by Jensen's Inequality was obtain the following result.

Proposition 2.2.

$$W[E_p(w)] = 1 - \sum_{j=k}^{m} a_j \cdot j\left[j + \frac{(\alpha_1 - 1)\beta_2}{\alpha_2 \beta_2}\right]^{-1} \leq E_p[R(\lambda)]$$

$$(2.5) \qquad \leq 1 - \sum_{j=k}^{m} a_j \cdot j\left[j + \frac{\alpha_1 \beta_2}{(\alpha_2 - 1)\beta_1}\right]^{-1} = R[E_p(\lambda)].$$

Hence, if the statistician has prior beliefs in terms of $E_p[R(\lambda)]$ (*i.e.*, not via the translation of information form θ_1 θ_2 to $R(\lambda)$), then he may use (2.5) to determine values of $\alpha_1, \alpha_2, \beta_1$ and β_2 which conforms his beliefs and which may be used in the prior distributions expressed by $g_1(\theta_1)$ and $g_2(\theta_2)$.

Note that the posterior distribution of θ_1 and θ_2 given $X_1, \cdots, X_{n1}$ and $Y_1, \cdots, Y_{n2}$ are $G(\alpha_1 + n_1, \beta_1 + n_1 \bar{X}_{n1})$ and $G(\alpha_2 + n_2, \beta_2 + n_2 \bar{Y}_{n2})$, respectively,

$$\text{where} \quad \bar{X}_{n1} = \frac{1}{n_1} \sum_{i=1}^{n1} X_i \quad \text{and} \quad \bar{Y}_{n2} = \frac{1}{n_2} \sum_{i=1}^{n2} Y_i \cdot$$

If we empoly the quardratic loss, the Bayes estimator of $R(\lambda)$ is easily obtained by (2.3). The Bayes estimator of $R(\lambda)$ given $\bar{X}_{n1}$ and $\bar{Y}_{n2}$ is as follows:

$$\tilde{R}_{k,m} = 1 - \sum_{j=k}^{m} a_j E\left(\frac{j}{\lambda+j} \mid \bar{X}_{n1}, \bar{Y}_{n2}\right)$$

$$= 1 - \left(\frac{\alpha_2 + n_2}{\alpha_1 + \alpha_2 + n_1 + n_2}\right) \left[\sum_{j=k}^{\delta 1} a(jT)^{\alpha 1 + n 1}\right.$$

$${}_2F_1(\alpha_1 + \alpha_2 + n_1 + n_2, \alpha_1 + \alpha_2 + n_1 + n_2 + 1, 1 - jT) + \sum_{j=\delta 2}^{m} a_j (jT)^{-\alpha 2 - n 2}$$

$$\left. {}_2F_1(\alpha_1 + \alpha_2 + n_1 + n_2, \alpha_2 + n_2 + 1, \alpha_1 + \alpha_2 + n_1 + n_2 + 1, 1 - (jT)^{-1})\right]$$

where $T = (\beta_1 + n_1 \bar{X}_{n1}) / (\beta_2 + n_2 \bar{Y}_{n2})$

$$\delta_1 = \min(m,\ T^{-1})$$
and $$\delta_2 = \max(k,\ T^{-1}+1)\ .$$
Posterior risk of $R_{k,m}$ is easily obtained as follows:

$$r(R_{k,m},\ \tilde{R}_{k,m}) = E[R_{k,m} - \tilde{R}_{k,m}]^2$$

$$= 1 + \frac{(\alpha_1+n_1)(\alpha_2+n_2)}{(\alpha_1+\alpha_2+n_1+n_2)(\alpha_1+\alpha_2+n_1+n_2+1)} [\sum_{j=k}^{\delta 1} a_j^2(jT)^{\alpha 1+n 1}$$

$${}_2F_1(\alpha_1+\alpha_2+n_1+n_2,\ \alpha_1+n_1,\ \alpha_1+\alpha_2+n_1+n_2+2,\ 1-jT)$$

$$+ \sum_{j=\delta 2}^{m} a_j^2(jT)^{-\alpha 2-n 2}\ {}_2F_1(\alpha_1+\alpha_2+n_1+\alpha_2,n_2+n_2+2,\ \alpha_1+\alpha_2+n_1+n_2+2,$$

$$1-(jT)^{-1})] + 2\frac{(\alpha_2+n_2)}{(\alpha_1+\alpha_2+n_1+n_2)}$$

$$[\sum_{j=k}^{\delta 1} a_j b_j (jT)^{\alpha 1+n 1}\ {}_2F_1(\alpha_1+\alpha_2+n_1+n_2,\ \alpha_1+n_1,\ \alpha_1+\alpha_2+n_1+n_2+1,$$

$$1-jT) + \sum_{j=\delta 2}^{m} a_j b_j (jT)^{-\alpha 2-n 2}\ {}_2F_1(\alpha_1+\alpha_2+n_1+n_2,\ \alpha_1+n_2+1,$$

$$\alpha_1+\alpha_2+n_1+n_2+1, 1-(jT)^{-1})] - [1-(\frac{\alpha_2+n_2}{\alpha_1+\alpha_2+n_1+n_2}) \{ \sum_{j=k}^{\delta 1} a_j(jT)^{\alpha 1+n 1}\ .$$

$${}_2F_1(\alpha_1+\alpha_2+n_1+n_2,\ \alpha_1+n_2+1,\ \alpha_1+\alpha_2+n_1+n_2+1,\ 1-(jT) + \sum_{j=\delta 2}^{m} a_j(jT)^{-\alpha 2-n 2}$$

$${}_2F_1(\alpha_1+\alpha_2+n_1+n_2,\ \alpha_2+n_2+1,\ \alpha_1+\alpha_2+n_1+n_2+1,\ 1-(jT)^{-1}\}]^2\ .$$

The predictive bounds of $\tilde{R}_{k,m}$ are obtained by the following inequalities.

Prosposition 2.3.

$$1 - \sum_{j=k}^{m} a_j \cdot j[j + \frac{(\alpha_1-n_1-1)(\beta_2-n_2\bar{Y}_{n2})}{(\alpha_2+n_2)(\beta_1+n_1\bar{X}_{n1})}]^{-1} \leq$$

$$\tilde{R}_{k,m} \leq 1 - \sum_{j-k}^{m} a_j \cdot j[j + \frac{(\alpha_1+n_1)(\beta_2+n_2\bar{Y}_{n2})}{(\alpha_2+n_2-1)(\beta_1+n_1\bar{X}_{n1})}]^{-1}\ .$$

Proof. It is easily obtained by the similar method as (2.5)
Next, we consider "vague" priors given by

$$g_1(\theta_1) \propto \frac{1}{\theta_1{}^{c1}}\ ,\ c_1 > 0$$

and

$$g_2(\theta_2) \propto \frac{1}{\theta_2{}^{c2}}\ ,\ c_2 > 0\ .$$

Note that the posterior distribution of θ_1 and θ_2 given $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n2}$ are $G(n_1-c_1+1,\ n_1\bar{X}_{n1})$ and $G(n_2-c_2+1,\ n_2\bar{Y}_{n2})$,respectively. Hence, if we employ the quardratic loss, the Bayes estimator of $R(\lambda)$ is directly obtained from (2.3) as follows:

The Bayes estimator of $R(\lambda)$ given $X_1, \cdots, X_{n1}$ and $Y_1, \cdots, Y_{n2}$ is

$$R^*_{k,m} = 1 - \left(\frac{n_2 - c_2 + 1}{n_1 + n_2 - c_1 - c_2 + 2}\right) \Big[\sum_{j=k}^{v_1} a_j (jV)^{n_1 - c_1 + 1} \cdot$$

$$_2F_1 (n_1 + n_2 - c_1 - c_2 + 2, n_1 - c_1 + 1, n_1 + n_2 - c_1 - c_2 + 3, 1 - jV)$$

$$+ \sum_{j=v_2}^{m} a_j (jV)^{-n_2 + c_2 - 1} \, _2F_1 (n_1 + n_2 - c_1 - c_2 + 2, \; n_2 - c_2 + 2,$$

$$n_1 + n_2 - c_1 - c_2 + 3, \; 1 - (jV)^{-1})\Big],$$

where $V = (n_1 \bar{X}_{n1}) / (n_2 \bar{Y}_{n2})$,

$v_1 = \min(m, V^{-1})$

and $v_2 = \max(k, V^{-1} + 1)$.

Inequalities on $R^*_{k,m}$ for this case are

$$1 - \sum_{j=k}^{m} a_j \cdot j \Big[j + \frac{(n_1 - c_1) n_2 \bar{Y}_{n2}}{(n_2 - c_2 + 1) n_1 \bar{X}_{n1}} \Big]^{-1}$$

$$\leq R^*_{k,m} \leq 1 - \sum_{j=k}^{m} a_j \cdot j \Big[j + \frac{(n_1 - c_1 + 1) n_2 \bar{Y}_{n2}}{(n_2 - c_2) n_1 \bar{X}_{n1}} \Big]^{-1}.$$

Finally, we note that for $\bar{Y}_{n2} / \bar{X}_{n1}$ and large values of n_1 and n_2 such that $n_1 \gg c_1$ and $n_2 \gg c_2$, $R^*_{k,m}$ may be approximated by

$$(2.7) \qquad 1 - \sum_{j=k}^{m} a_j \cdot j \Big[j + \frac{\bar{Y}_{n2}}{\bar{X}_{n1}} \Big]^{-1}$$

The MLE of (θ_1, θ_2) is given by $\hat{\theta}_1 = 1/\bar{X}_{n1}$, $\hat{\theta}_2 = 1/\bar{Y}_{n2}$. Using the invariance property of MLE and the expression (2.1), the MLE $\hat{R}_{k,m}$ of $R(\lambda)$ is as follows:

$$(2.8) \qquad \hat{R}_{k,m} = 1 - \sum_{j=k}^{m} a_j \cdot j \Big[j + \frac{\bar{Y}_{n2}}{\bar{X}_{n1}} \Big]^{-1}$$

In large samples, we know not that the choices of constants c_1 and c_2 are not very crucial.

III. ASYMPTOTIC DISTRIBUTION

Here we investigate the asymptotic distribution of the Bayes estimator, $R^*_{k,m}$ given in (2.6). Due to the complexity of the expression, it is rather tedious to drive the limiting distribution of $R^*_{k,m}$ directly form (2.6). To circumvent this difficulty, we consider the asymptotic properties of the MLE of $R_{k,m}$ and then establish its asymptotic equivalence with $R^*_{k,m}$.

We write $R^*_{(n)k,m}$ for Bayes estimator given in (2.6) and $\hat{R}_{(n)k,m}$ for MLE given (2.8) where $n = n_1 + n_2$ is the combined sample size.

Proposition 3.1. Let $n \to \infty$ such that $n_1/n \to f,\ 0 < f < 1$.

Then $\quad n^{\frac{1}{2}}(R^*_{(n)k,m} - R_{k,m}) \xrightarrow{L} N(0, \sigma^2_{k,m})$

where $\quad \sigma^2_{k,m} = [f(1-f)]^{-1} \cdot [\sum_{j=k}^{m} a_j \frac{j}{\lambda + j}]^2 \cdot [\sum_{j=k}^{m} 1 - \frac{j}{\lambda + j}]^2$

and $\xrightarrow{L}$ denotes conbergence in distribution.

Proof. From the Theorem 3.1 in Bhattacharyya and Johnson [1], we know that $n^{\frac{1}{2}}(\hat{R}_{(n)k,m} - R_{k,m}) \xrightarrow{L} N(0, \sigma^2_{k,m})$.

$R^*_{(n)k,m}$'s asymptotic equivalence with MLE can be established by (2.7) and (2.8).

IV. EMPIRICAL COMPARISON IN MODERATE SAMPLES

In Section 2, for vague priors, it was shown that in large samples, the choices of constants c_1 and c_2 are not very crucial. In this section for gamma conjugate prior distributions, Bayes estimates of $R_{k,m}$ are compared with one another. Bayes estimates of $R_{k,m}$ are obtained in a moderate sample $n_1 = 10, n_2 = 10$ for the 2 out of 3 and 1 out of 4 systems with $\lambda = 0.1, 0.5, 1, 2, 10$ (actually, with (θ_1, θ_2) such that (1, 10) (1, 2), (1,1), (2, 1), (10, 1)).

The results on the Bayes estimates of $R_{k,m}$ appear in the tables. Note that the values of (Min./Max.) for fixed (α_1, β_1) and varying (α_2, β_2) are close to 1. Similarly, the values of (Min./Max.) for fixed (α_2, β_2) and varying (α_1, β_1) are near to 1, except for the case $(\theta_1, \theta_2) = (1, 10)$. Bayes estimates of $R_{k,m}$ of 1 out of 4 system are greater them those of 2 out of 3 system. It is coincides with our common sense. It is known that Bayes estimates of $R_{k,m}$ are increase as λ increase for all 2 out of 3 and 1 out of 4 systems. Also, we know that Bayes estimates of $R_{k,m}$ have a trend such that for fixed (α_1, β_1), it is decrease as (α_2, β_2) increase and for fixed (α_2, β_2) it is increase as (α_1, β_1) increase.

Table 1. Bayes Estimates of $R_{k,m}$

$(n_1, n_2) = (10, 10), (\theta_1, \theta_2) = (1, 1), (k, m) = (1, 4)$

(α_2, β_2) \ (α_1, β_1)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	Min./Max.
(1, 1)	0.7857	0.7892	0.7884	0.7895	0.7904	0.9941
(2, 2)	0.7854	0.7869	0.7881	0.7892	0.7901	0.9941
(3, 3)	0.7851	0.7866	0.7878	0.7889	0.7898	0.9940
(4, 4)	0.7849	0.7863	0.7876	0.7887	0.7896	0.9940
(5, 5)	0.7847	0.7862	0.7864	0.7885	0.7894	0.9940
Min./Max.	0.9987	0.9987	0.9987	0.9987	0.9987	

Table 2. Bayes Estimates of $R_{k,m}$

(n_1, n_2) (10, 10), $(\theta_1, \theta_2) = (1, 1)$, $(k, m) = (2,3)$

(α_1, β_1) / (α_2, β_2)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	Min./Max.
(1, 1)	0.5027	0.5037	0.5045	0.5052	0.5056	0.9943
(2, 2)	0.5017	0.5026	0.5037	0.5041	0.5046	0.9943
(3, 3)	0.5507	0.5016	0.5024	0.5031	0.5036	0.9942
(4, 4)	0.4999	0.5008	0.5016	0.5023	0.5028	0.9942
(5, 5)	0.4991	0.5001	0.5009	0.5016	0.5020	0.9942
Min./Max.	0.9928	0.9929	0.9929	0.9929	0.9929	

Table 3. Bayes Estimates of $R_{k,m}$

$(n_1,n_2)=(10,10),(\theta_1,\theta_2) = (1,2),(k,m) = (1,4)$

(α_1, β_1) / (α_2, β_2)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	Min./Max.
(2, 1)	0.5916	0.5927	0.5936	0.5945	0.5953	0.9938
(4, 2)	0.5899	0.5911	0.5921	0.5930	0.5936	0.9938
(6, 3)	0.5886	0.5898	0.5908	0.5917	0.5923	0.9938
(8, 4)	0.5876	0.5888	0.5898	0.5907	0.5913	0.9937
(10, 5)	0.5869	0.5880	0.5890	0.5898	0.5906	0.9937
Min./Max.	0.9921	0.9921	0.9921	0.9921	0.9921	

Table 4. Bayes Estimates of $R_{k,m}$

$(n_1,n_2) = (10,10), (\theta_1,\theta_2) = (2,1), (k,m) = (1,4)$

(α_1, β_1) / (α_2, β_2)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	Min./Max.
(2, 1)	0.3230	0.3234	0.3238	0.3242	0.3244	0.9957
(4, 2)	0.3210	0.3215	0.3219	0.3222	0.3224	0.9957
(6, 3)	0.3195	0.3200	0.3204	0.3207	0.3209	0.9956
(8, 4)	0.3183	0.3188	0.3192	0.3195	0.3197	0.9956
(10,5)	0.3174	0.3178	0.3182	0.3186	0.3188	0.9956
Min./Max.	0.9827	0.9827	0.9827	0.9827	0.9827	

Table 5. Bayes Estimates of $R_{k,m}$

$(n_1, n_2) = (10,10)$, $(\theta_1, \theta_2) = (2,1)$, $(k,m) = (1,4)$

(α_1, β_1) / (α_2, β_2)	(2,1)	(4,2)	(6,3)	(8,4)	(10,5)	Min./Max.
(1, 1)	0.9187	0.9204	0.9218	0.9929	0.9237	0.9946
(2, 2)	0.9188	0.9206	0.9219	0.9230	0.9239	0.9945
(3, 3)	0.9189	0.9207	0.9221	0.9231	0.9240	0.9945
(4, 4)	0.9189	0.9208	0.9221	0.9232	0.9240	0.9945
(5, 5)	0.9190	0.9208	0.9222	0.9233	0.9421	0.9945
Min./Max.	0.9997	0.9996	0.9996	0.9996	0.9996	

Table 6. Bayes Estimates of $R_{k,m}$

$(n_1, n_2) = (10,10)$, $(\theta_1, \theta_2) = (2,1)$, $(k,m) = (2,3)$

(α_1, β_1) / (α_2, β_2)	(2, 1)	(4,2)	(6,3)	(8,4)	(10,5)	Min./Max.
(1, 1)	0.6930	0.6949	0.6965	0.6977	0.6986	0.9920
(2, 2)	).6924	0.6943	0.6958	0.6971	0.6980	0.9920
(3, 3)	0.6919	0.6938	0.6953	0.6965	0.6975	0.9920
(4, 4)	0.6913	0.6933	0.6949	0.6961	0.6969	0.9920
(5, 5)	0.6909	0.6929	0.6945	0.6957	0.6965	0.9920
Min./Max.	0.9970	0.9971	0.9971	0.9971	0.9970	

Table 7. Bayes Estimates of $R_{k,m}$

$(n_1, n_2) = (10,10)$, $(\theta_1, \theta_2) = (10,1)$, $(k,m) = (1,4)$

(α_1, β_1) / (α_2, β_2)	(1, 0.1)	(2, 0.2)	(3, 0.3)	(4, 0.4)	(5, 0.5)	Min./Max.
(1, 1)	0.9971	0.9972	0.9973	0.9974	0.9975	0.9996
(2, 2)	0.9966	0.9967	0.9968	0.9969	0.9969	0.9997
(3, 3)	0.9959	0.9961	0.9962	0.9963	0.9963	0.9996
(4, 4)	0.9952	0.9953	0.9955	0.9956	0.9956	0.9996
(5, 5)	0.9944	0.9946	0.9948	0.9949	0.9949	0.9995
Min./Max.	0.9973	0.9974	0.9975	0.9975	0.9974	

Table 8. Bayes Estimates of $R_{k,m}$

$(n_1, n_2) = (10,10), (\theta_1,\theta_2) = (10,1), (k,m) = (2,3)$

(α_1,β_1) \ (α_2,β_2)	(1, 0.1)	(2, 0.2)	(3, 0.3)	(4, 0.4)	(5, 0.5)	Min./Max.
(1, 1)	0.9469	0.9476	0.9482	0.9487	0.9491	0.9977
(2, 2)	0.9408	0.9416	0.9422	0.9427	0.9431	0.9976
(3, 3)	0.9346	0.9354	0.9361	0.9367	0.9371	0.9973
(4, 4)	0.9285	0.9354	0.9361	0.9367	0.9371	0.9973
(5, 5)	0.9224	0.9233	0.9241	0.9247	0.9252	0.9970
Min./Max.	0.9743	0.9744	0.9746	0.9747	0.9748	

Table 9. Bayes Estimates of $R_{k,m}$

$(n_1,n_2) = (10,10), (\theta_1,\theta_2) = (1,10), (k,m) = (2,3)$

(α_1,β_1) \ (α_2,β_2)	(1, 1)	(2, 2)	(3, 3)	(4, 4)	(5, 5)	Min./Max.
(1, 0.1)	0.0904	0.0965	0.1035	0.1107	0.1176	0.7687
(2, 0.2)	0.0895	0.0959	0.1031	0.1104	0.1173	0.7630
(3, 0.3)	0.0887	0.0953	0.1027	0.1101	0.1170	0.7581
(4, 0.4)	0.0880	0.0949	0.1025	0.1099	0.1168	0.7534
(5, 0.5)	0.0875	0.0946	0.1023	0.1097	0.1166	0.7504
Min./Max.	0.9679	0.9803	0.9884	0.9907	0.9915	

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