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Statistical Inferences for Bivariare Exponential Distribution in Reliability and Life Testing Problems

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ABSTRACT

In this paper, statistical estimation of the parameters of the bivariate exponential distribution are studied. Bayes estimators of the parameters are obtained and compared with the maximum likelihood estimators which are introduced by Freund. We know that the method of moments estimators coincide with the maximum likelihood estimators and Bayes estimators are more efficient than the maximum likelihood estimators in moderate samples. The asymptotic distributions of the maximum likelihood estimators and the estimator of mean time to system failure are obtained.

1. Introduction

In problem of life testing and reliability analysis, the exponential distribution playes a central role as useful statistical model. In a system consisting of complex multicomponents linked in series or parallel, common assumptions are that the component lifetimes are independent and exponentially distributed. But occasionally, independent assumption is not applicable in the practical situation. In such cases Freund (1961), Marshall and Olkin (1967), Downton (1970) and Hawkes (1972) studied the estimation problems of the reliability.

In this paper we consider the bivariate exponential distribution model introduced by Freund (1961). The model denotes the parallel system with two components which functions when one of components are failed. Let X_1 and X_2 be random variables to denote the lifetimes of two components A and B, respectively. The lifetimes of components distribute independently and exponentially with failure rates α and β, respectively, until first failure time, $\min(X_1, X_2)$. After the time $\min(X_1, X_2)$, the unfailed component's lifetime distribute exponentially with a new failure rate α' or β' according to B or A failed. This model may realistically represent systems in which the dependence between X_1 and X_2 in such that the failure of one component puts additional burden on the remaining one or, alternatively the failure of one component may relieve somewhat the burden on the other. For example, this model is applicable to describe such situations as the failure of paired organs such as lungs, kidneys

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and eyes. In certain types of diseases it is possible that the simultaneous failure of the paired organ is very rare.

When the failed components are not replaced, Freund (1961) showed that the joint density of (X_1, X_2) has

$$f(x_1, x_2) = \begin{cases} \alpha\beta' \exp\{-\beta' x_2 - (\alpha+\beta-\beta')x_1\}, & 0 < x_1 < x_2, \\ \beta\alpha' \exp\{-\alpha' x_1 - (\alpha+\beta-\alpha')x_2\}, & 0 < x_2 < x_1, \end{cases} \tag{1.1}$$

where α, β, α' and $\beta' > 0$. He obtained the moment generating function and the maximum likelihood estimators (MLEs) of the parameters of this model. The marginal distributions of (1.1) are not exponential.

Block and Basu (1974) showed that a three-parameter subfamily of Freund's distributions corresponds to the absolutely continuous component of the bivariate exponential distribution derived by Marshall and Olkin(1967).

In section 2 we obtain the properties of MLEs and the method of moments estimators (MMEs) of the parameters. The asymptotic distributions of MLEs and the mean time to failure (MTTF) of parallel system with two components are obtained under the Freund's model.

In section 3 we obtain Bayes estimators of the parameters under the quadratic loss function.

In section 4 we compare the efficiencies of MLEs with Bayes estimators in a moderate sample size through Monte Carlo simulation. Also we obtain the mean squared error (MSE) and bias of estimated MTTF.

2. Asymptotic Properties of MLEs and MMEs

Consider a parallel system with two components whose lifetimes are X_1 and X_2. Suppose that the joint density of (X_1, X_2) is (1.1). In a random sample of size n from a population (1.1) we assume that components of type A fail first r times and components of type B fail first $n-r$ times. The likelihood function is expressed as follows:

$$L(\alpha, \beta, \alpha', \beta')$$

$$= (\alpha\beta')^r(\beta\alpha')^{n-r} \exp\{-\beta'\textstyle\sum x_2 - (\alpha+\beta-\beta')\sum x_1 - \alpha'\sum' x_1 - (\alpha+\beta-\alpha')$$

$$\textstyle\sum' x_2\}, \quad 0 < r < n, \tag{2.1}$$

where $\sum x_1 =$ sum of the lifetimes of the components A which failed first and $\sum x_2 =$ sum of the lifetimes of the corresponding components B, $\sum' x_2 =$ sum of the lifetimes of the components B which failed first and $\sum' x_1 =$ sum of the lifetimes of the corresponding components A.

Based on the above likelihood function, Freund showed that

$$\hat{\alpha} = \frac{r}{\sum x_1 + \sum' x_2},$$

$$\hat{\beta} = \frac{n - r}{\sum x_1 + \sum' x_2} ,$$

$$\hat{\alpha}' = \frac{n - r}{\sum' x_1 - \sum' x_2} \tag{2.2}$$

and

$$\hat{\beta}' = \frac{r}{\sum x_2 - \sum x_1}$$

He also obtained means and variances of MLEs for the fixed r.

Now, consider the asymptotic properties of the MLEs $\hat{\alpha}, \hat{\beta}, \hat{\alpha}'$ and $\hat{\beta}'$. We write $\underline{\hat{\theta}}_n = (\hat{\alpha}, \hat{\beta}, \hat{\alpha}', \hat{\beta}')'$ for the MLEs of $\underline{\theta} = (\alpha, \beta, \alpha', \beta')'$. Then we get the following theorem:

Theorem 1. Let Q = (the second derivatives in (2.1)).

Then,

(a) The random vector $\underline{\hat{\theta}}_n \to \underline{\theta}$ as $n \to \infty$ with probability 1.

(b) The random vector $\sqrt{n}(\underline{\hat{\theta}}_n - \underline{\theta})$ has asymptotically the multivariate normal distribution $N(\underline{0}, \Sigma)$ where $\Sigma^{-1} = E\left\{-n^{-1} Q\right\}$ is the information matrix given by

$$\Sigma^{-1} = \text{diag}\left(\frac{1}{(\alpha+\beta)\alpha}, \frac{1}{(\alpha+\beta)\beta}, \frac{\beta}{(\alpha+\beta)\alpha'^2}, \frac{\alpha}{(\alpha+\beta)\beta'^2} \right). \tag{2.3}$$

Proof. (a) For sufficiently large n, $\underline{\hat{\theta}}_n \to \underline{\theta}$ with probability one by the consistency property of MLE (Rao(1973), page 365).

(b) From (2.1) $Q = \text{diag}\left(\frac{-r}{\alpha^2}, \frac{-(n-r)}{\beta^2}, \frac{-(n-r)}{\alpha'^2}, \frac{-r}{\beta'^2} \right)$,

and r is random variable having binomial distribution with parameters n and $\frac{\alpha}{\alpha+\beta}$. Computing $E(Q)$, we can easily obtain (2.3.). The likelihood function (2.1) satisfies the Cramer conditions (Rao (1973), page 361) for asymptotic normality. Hence we obtain part (b).

Next, we consider the MMEs $\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha}'$ and $\tilde{\beta}'$ of the parameters α, β, α' and β'. Set $\sum x_1 + \sum' x_1$, $\sum x_2 + \sum' x_2$, $\sum x_1 + \sum' x_2$ and r equal to their respective expected values. Then we obtain the following equations:

$$\Sigma x_1 + \Sigma' x_1 = n \left\{ \frac{\alpha' + \beta}{\alpha'(\alpha+\beta)} \right\} ,$$

$$\Sigma x_2 + \Sigma' x_2 = n \left\{ \frac{\beta' + \alpha}{\beta'(\alpha+\beta)} \right\} ,$$

$$\Sigma x_1 + \Sigma' x_2 = \frac{n}{\alpha + \beta}$$

and

$$r = \frac{n\alpha}{\alpha + \beta} .$$

From the above equations we obtain MMEs as follows:

$$\tilde{\alpha} = \frac{r}{\Sigma x_1 + \Sigma' x_2} ,$$

$$\tilde{\beta} = \frac{n - r}{\Sigma x_1 + \Sigma' x_2} ,$$

$$\tilde{\alpha}' = \frac{n - r}{\Sigma' x_1 - \Sigma' x_2} \qquad (2.4)$$

and

$$\tilde{\beta}' = \frac{r}{\Sigma x_2 - \Sigma x_1} .$$

Note that the MMEs in (2.4) coincide with the MLEs in (2.2). The asymptotic properties of MMEs coincide with the asymptotic properties of MLEs.

On the other hand, we consider the MTTF for parallel system with two components having the joint density (1.1). The system reliability is $\bar{F}(t) = P\{\max(X_1, X_2) > t\}$ as follows:

i) For the case $\alpha + \beta \neq \alpha'$ and $\alpha + \beta \neq \beta'$,

$$\bar{F}(t) = (1 - \frac{\alpha}{\alpha + \beta - \beta'} - \frac{\beta}{\alpha + \beta - \alpha'}) e^{-(\alpha + \beta) t}$$

$$+ \frac{\alpha}{\alpha + \beta - \beta'} e^{-\beta' t} + \frac{\beta}{\alpha + \beta - \alpha'} e^{-\alpha' t} . \qquad (2.5)$$

ii) For the case $\alpha + \beta \neq \alpha'$ and $\alpha + \beta = \beta'$,

$$\bar{F}(t) = (1 - \frac{\beta}{\beta' - \alpha'}) e^{-\beta' t} + \frac{\beta}{\beta' - \alpha'} e^{-\alpha' t} + \alpha t e^{-\beta' t} . \qquad (2.6)$$

iii) For the case $\alpha + \beta = \alpha'$ and $\alpha + \beta \neq \beta'$,

$$\bar{F}(t) = (1 - \frac{\alpha}{\alpha' - \beta'}) e^{-\alpha' t} + \frac{\alpha}{\alpha' - \beta'} e^{-\beta' t} + \beta t e^{-\alpha' t} . \qquad (2.7)$$

iv) For the case $\alpha + \beta = \alpha' = \beta'$

$$\bar{F}(t) = e^{-\alpha' t} + \alpha' t e^{-\alpha' t} . \qquad (2.8)$$

Hence the MTTF based on $\bar{F}(t)$ is

$$\mu = \int_0^\infty \bar{F}(t)\, dt$$

$$= \frac{\alpha\alpha' + \beta\beta' + \alpha'\beta'}{\alpha'\beta'(\alpha + \beta)} , \qquad (2.9)$$

and μ is estimated by

$$\hat{\mu} = \frac{\hat{\alpha}\hat{\alpha}' + \hat{\beta}\hat{\beta}' + \hat{\alpha}'\hat{\beta}'}{\hat{\alpha}'\hat{\beta}'(\hat{\alpha} + \hat{\beta})} \tag{2.10}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\alpha}'$ and $\hat{\beta}'$ are the MLEs of the parameters α, β, α' and β', respectively.

Theorem 2. For MLE $\hat{\mu}$ of μ,

(a) $\hat{\mu}$ is an unbiased estimator of μ.

(b) $\sqrt{n}(\hat{\mu} - \mu)$ has asymptotically the normal distribution $N(0, \sigma^2)$ where

$$\sigma^2 = 2\left(\frac{1}{\alpha + \beta} + \frac{1}{\alpha'} + \frac{1}{\beta'}\right). \tag{2.11}$$

Proof. (a) Using $\hat{\alpha}$, $\hat{\beta}$, $\hat{\alpha}'$ and $\hat{\beta}'$ given in (2.2), we have

$$\hat{\mu} = \frac{1}{n}\left\{(\Sigma x_1 + \Sigma' x_2) + (\Sigma' x_1 - \Sigma' x_2) + (\Sigma x_2 - \Sigma x_1)\right\}.$$

By the definitions of Freund's model, r is random variable having binomial distribution with parameters n and $\frac{\alpha}{\alpha + \beta}$, $\Sigma x_1 + \Sigma' x_2$ is random variable having gamma distribution with parameters n and $\frac{1}{\alpha + \beta}$, $\Sigma' x_1 - \Sigma' x_2$ is random variable having gamma distribution with parameters $n - r$ and $\frac{1}{\alpha'}$, and $\Sigma x_2 - \Sigma x_1$ is random variable having gamma distribution with parameters r and $\frac{1}{\beta'}$. Hence we obtain

$$E(\hat{\mu}) = E\left\{E(\hat{\mu} \mid r)\right\} = \frac{\alpha\alpha' + \beta\beta' + \alpha'\beta'}{\alpha'\beta'(\alpha + \beta)} = \mu.$$

(b) Since $\hat{\mu}$ is a totally differential function of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\alpha}'$ and $\hat{\beta}'$ by Theorem 6a. 2(ii) on page 387 of Rao (1973), $\sqrt{n}(\hat{\mu} - \mu)$ is asymptotically normally distributed with mean zero and variance σ^2, where $\sigma^2 = \Sigma\Sigma\sigma_{ij} x$ (corresponding second derivative in (2.9)) and σ_{ij} = the (i, j) element of Σ given by Theorem 1. After the tedious calculation, we can obtain (2.11).

3. Bayes Estimators

In this section, using Bayesian approach, we consider the estimation of the parameters α, β, α' and β' under the quadratic loss function $(\hat{\underline{\theta}}_n - \underline{\theta})'(\hat{\underline{\theta}}_n - \theta)$ where (X_1, X_2) has a joint density (1.1).

Consider a general class of vague prior distribution given by

$$g(\alpha, \beta, \alpha', \beta') \propto \frac{1}{\alpha^{c_1} \beta^{c_2} \alpha'^{c_3} \beta'^{c_4}} \tag{3.1}$$

where c_1, c_2, c_3 and c_4 are arbitrary positive constants. From (2.1) we have the likelihood function as follows.

$$L(\alpha, \beta, \alpha', \beta' \mid \underline{x}_1, \underline{x}_2)$$
$$= (\alpha\beta')^r (\beta\alpha')^{n-r} \exp\left\{-(\alpha + \beta - \beta')\Sigma x_1 - \beta'\Sigma x_2 - \alpha'\Sigma' x_1 - (\alpha + \beta - \alpha')\Sigma' x_2\right\}. \tag{3.2}$$

The posterior distribution of α, β, α' and β' is given by

$$\Pi\,(\alpha,\beta,\alpha',\beta' \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2)$$

$$= \frac{k}{\alpha^{c_1}\,\beta^{c_2}\,\alpha'^{c_3}\,\beta'^{c_4}}\; L\,(\alpha,\beta,\alpha',\beta' \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2) \tag{3.3}$$

where

$$k^{-1} = \frac{\Gamma(r-c_1+1)\;\Gamma(n-r-c_2+1)}{(\Sigma x_1+\Sigma' x_2)^{r-c_1+1}\,(\Sigma x_1+\Sigma' x_2)^{n-r-c_2+1}}$$

$$\cdot\; \frac{\Gamma(n-r-c_3+1)\;\Gamma(r-c_4+1)}{(\Sigma' x_1-\Sigma' x_2)^{n-r-c_3+1}\,(\Sigma x_2-\Sigma x_1)^{r-c_4+1}}\;.$$

Then the marginal posterior of α is

$$\Pi_1\,(\alpha \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2)$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty \Pi\,(\alpha,\beta,\alpha',\beta')\; d\beta\, d\alpha'\, d\beta' \tag{3.4}$$

$$= \frac{(\Sigma x_1+\Sigma' x_2)^{r-c_1+1}}{\Gamma(r-c_1+1)}\;\alpha^{r-c_1}\exp\left\{-(\Sigma x_1+\Sigma' x_2)\,\alpha\right\},\quad \alpha>0\,.$$

Similarly,

$$\Pi_2\,(\beta \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2) \tag{3.5}$$

$$= \frac{(\Sigma x_1+\Sigma' x_2)^{n-r-c_2+1}}{\Gamma(n-r-c_2+1)}\;\beta^{n-r-c_2},\exp\left\{-(\Sigma x_1+\Sigma' x_2)\,\beta\right\},\quad \beta>0\,,$$

$$\Pi_3\,(\alpha' \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2) \tag{3.6}$$

$$= \frac{(\Sigma' x_1-\Sigma' x_2)^{n-r-c_3+1}}{\Gamma(n-r-c_3+1)}\;\alpha'^{\,n-r-c_3}\exp\left\{-(\Sigma' x_1-\Sigma' x_2)\,\alpha'\right\},\quad \alpha'>0$$

and

$$\Pi_4\,(\beta' \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2) \tag{3.7}$$

$$= \frac{(\Sigma x_2-\Sigma x_1)^{r-c_4+1}}{\Gamma(r-c_4+1)}\;\beta'^{\,r-c_4}\exp\left\{-(\Sigma x_2-\Sigma x_1)\,\beta'\right\},\quad \beta'>0\,.$$

From (3.4) we have that Bayes estimator is given by

$$\alpha^* = E\,(\alpha \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2) = \int_0^\infty \alpha\,\Pi_1\,(\alpha \mid \underset{\sim}{x}_1, \underset{\sim}{x}_2)\, d\alpha = \frac{r-c_1+1}{\Sigma x_1+\Sigma' x_2}\;. \tag{3.8}$$

Similarly, from (3.5) to (3.7), we have

$$\beta^* = \frac{n-r-c_2+1}{\Sigma x_1 + \Sigma' x_2}, \tag{3.9}$$

$$\alpha'^* = \frac{n-r-c_3+1}{\Sigma' x_1 - \Sigma' x_2} \tag{3.10}$$

and

$$\beta'^* = \frac{r-c_4+1}{\Sigma x_2 - \Sigma x_1}. \tag{3.11}$$

For the case $c_1 = c_2 = c_3 = c_4 = 1$,

$$\alpha^* = \hat{\alpha},\ \beta^* = \hat{\beta},\ \alpha'^* = \hat{\alpha}' \text{ and } \beta'^* = \hat{\beta}',$$

which are coincided with MLEs of (2.2).

4. Empirical Comparison

We can obtain the biases and the mean squared errors(MSEs) of Bayes estimators α^*, β^*, α'^* and β'^*, respectively, as follows:

1. $\text{Bias}(\alpha^*) = \frac{1}{n-1}\{\alpha - (c_1-1)(\alpha+\beta)\},\ n > 1,$

$$\text{MSE}(\alpha^*) = \frac{1}{(n-1)(n-2)}\{\alpha((n+2)\alpha+n\beta) + (c_1-1)(\alpha+\beta)((c_1-1)(\alpha+\beta)-4\alpha)\},\ n > 2. \tag{4.1}$$

2. $\text{Bias}(\beta^*) = \frac{1}{n-1}\{\beta - (c_2-1)(\alpha+\beta)\},\ n > 1,$

$$\text{MSE}(\beta^*) = \frac{1}{(n-1)(n-2)}\{\beta((n+2)\beta+\alpha n) + (c_2-1)(\alpha+\beta)((c_2-1)(\alpha+\beta)-4\beta)\},\ n > 2. \tag{4.2}$$

3. For fixed r,

$$\text{Bias}(\alpha'^*) = \frac{2-c_3}{n-r-1}\alpha',\ r < n-1, \tag{4.3}$$

$$\text{MSE}(\alpha'^*) = \frac{(n-r-c_3+1)^2}{(n-r-1)^2(n-r-2)}\alpha'^2 + \{\text{Bias}(\alpha'^*)\}^2,\ r < n-2.$$

4. For fixed r, (4.4)

$$\text{Bias}(\beta'^*) = \frac{2-c_4}{r-1}\beta',\ r > 1,$$

$$\text{MSE}(\beta'^*) = \frac{(r-c_4+1)^2}{(r-1)^2(r-2)}\beta'^2 + \{\text{Bias}(\beta'^*)\}^2,\ r > 2. \tag{4.4}$$

In this paper, the efficiency of the estimators is measured in terms of the ratio of the sum of the Cramer-Rao lower bounds of the individual estimators to the sum of the MSEs of the individual estimators. It is given by

$$\text{Eff.} = tr(I_n^{-1})/\Sigma\text{MSE}$$

where $I_n^{-1} = n^{-1}\Sigma$.

Estimates of the MSEs are obtained from 400 simulated samples of size 20 and 40, respectively. In each situation generating two dependent exponential random variables of Freund's distribution, we use the method proposed by Friday and Patil (1977). Let us explain it in detail.

Let random variables Y_1 and Y_2 be independent with standard exponential distributions and random variables X_1 and X_2 define as follows:

$$X_1 = \begin{cases} Y_1\alpha^{-1} & \text{if } \beta Y_1 < \alpha Y_2, \\ Y_1(\alpha')^{-1}-(\alpha-\alpha')Y_2(\alpha'\beta)^{-1} & \text{if } \beta Y_1 > \alpha Y_2, \end{cases}$$

and

$$X_2 = \begin{cases} Y_2(\beta')^{-1}-(\beta-\beta')Y_1(\alpha\beta')^{-1} & \text{if } \beta Y_1 < \alpha Y_2, \\ Y_2\beta^{-1} & \text{if } \beta Y_1 > \alpha Y_2, \end{cases}$$

where α, β, α' and $\beta' > 0$.

Then X_1 and X_2 have the bivariate exponential distribution given by (1.1)

In the samples of size 20 and 40 efficiences of estimators are given in the following tables:

Table 1. Comparison of the Efficiencies of the Estimators
($c_1 = c_2 = c_3 = c_4 = 2$, $\alpha = 1$)

Parameters			Efficiencies		
β	α'	β'	Estimators \ n	20	40
0.5	1	1	MLE Bayes E.	0.53 0.75	0.76 0.89
0.5	1.5	1	MLE Bayes E.	0.48 0.72	0.73 0.87
1	2	1	MLE Bayes E.	0.45 0.69	0.71 0.86
1	1	1.5	MLE Bayes E.	0.61 0.79	0.80 0.90
1	1.5	1.5	MLE Bayes E.	0.59 0.78	0.79 0.89

1	2	1.5	MLE Bayes E.	0.58 0.77	0.78 0.89
3	3	4	MLE Bayes E.	0.42 0.69	0.63 0.82
3	4	4	MLE Bayes E.	0.44 0.70	0.64 0.83
4	3	5	MLE Bayes E.	0.35 0.65	0.61 0.83

The results are as follows;

Bayes estimators are more efficient than MLEs for $\alpha = 1$, $c_1 = c_2 = c_3 = c_4 = 2$ and various value of other parameters.

Table 2. MSE and Bias of Estimated MTTF ($\alpha = 1$)

n			10		20		40	
Parameters								
β	α'	β'	MSE	BIAS	MSE	BIAS	MSE	BIAS
0.5	0.5	1	0.121	0.005	0.061	0.004	0.027	0.002
1	2	1	0.115	0.014	0.060	0.012	0.028	0.011
1	1	1.5	0.100	0.013	0.048	0.010	0.024	0.005
1	1.5	1.5	0.063	0.016	0.035	0.011	0.018	0.001
1	2	1.5	0.055	0.014	0.029	0.013	0.016	0.004

MSE and Bias of estimated MTTF decrease as sample size n increases, which verify paper's result.

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