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Optimal Replacement Policy for a System Subject to Shocks

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ABSTRACT

A replacement policy for a system subject to shocks where each shock increases the running cost is considered. The shocks arrive to the system according to a nonhomogeneous Poisson process. Optimal replacement policy to minimize the long-run expected cost rate is obtained and some numerical examples are given.

1. Introduction

We consider a system subject to shocks, where each shock reduces the effectiveness of the system and makes it more expensive to run the system. The word shock in this paper has broad interpretations. In some situations, the intensity rate of shocks may increase with the age of the system. For example, consider a large system composed of many components where a shock is interpreted as the failure of one of the components. Although the system continues to operate after a shock, it does so under more stress due to increased loading of the other components and is therefore more susceptible to shocks.

Replacement policies for systems subject to shocks are usually based on the cumulative damage level. Taylor (1975) derived an optimal replacement policy for a system where shocks occur in accordance with a Poisson process. Each shock causes a random amount of damage, the damage accumulates additively and the accumuluted damage is observable. Zuckerman (1978) considered a failure model where the shock process is an increasing one with stationary independent increments. Attia and Brockwell(1984) studied an optimal replacement policy with continuously varying observable shocks.

On the other hand, Boland and Proschan (1983) studied the shock process in view of the costs and proposed the periodic replacement policy when the running cost of the system increases with the number of shocks arrived.

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In this paper, we perform a cost analysis on a system subject to shocks. Our approach is similar to that of Boland and Proschan (1983). We assume that the nomral running cost of the new system is c units per unit time, and that each shock to the system increases the running cost by c_1 unit per unit time. The cost of replacing the whole system is $c_0 (c_0 > c_1)$. The system is replaced at the first shock after age T which is assumed to be observable. We attempt to find the value of T^* that minimizes the long-run expected cost per unit time.

Let $r(t)$ denote the intensity rate at which the system of age t is subject to shocks. We assume that $r(t)$ is a continuous positive function for $t > 0$. Let $R(t) = \int_0^t r(s)\,ds$ denote the mean value function. We assume that the number of shocks arriving in the age interval $[0, t]$ of the system follows a nonhomogeneous Poisson process with the intensity function r(t).

2. Analysis

Let $N(t)$ be the number of shocks arrived in the age interval $[0, t]$ and Y_i, $i \geq 1$, denote the time at which the i^{th} shock occurred. Suppose that the system is subject to k shocks during the interval $[0, T]$. Then the running cost in the interval $[0, Y_{k+1}]$ is

$$cY_{k+1} + c_1(Y_2 - Y_1) + \cdots + c_1 k (Y_{k+1} - Y_k) = (c + c_1 k)\, Y_{k+1} - c_1 \sum_1^k Y_i .$$

This leads us to the following lemma.

Lemma 1. The conditional expected cost of running the system until the replacement of the system given $\{N(T) = k\}$ is

$$(c + c_1 k)\, M(T) + cT + c_1 k \int_0^T R(s)\, ds / R(T),$$
$$\text{for } k = 0, 1, \cdots,$$

where $M(T) = \int_T^\infty (1 - F(z))\, dz / (1 - F(T))$ (mean residual life function) and $F(t) = 1 - \exp(-R(t))$, $t \geq 0$.

Proof. First note that $E(Y_{N(T)+1} - T \mid N(T) = k) = M(T)$ (see Muth (1977)). For fixed k, let $Z_i = R(Y_i)$ for $i = 1, 2, \ldots, k$, Since the shock process is a nonhomogeneous Poisson process with intensite rate $r(t)$, we know that given k shocks in the interval $[0, T)$, the random variables $Z_1, \ldots,$ and Z_k are distributed as the order statistics in the sample of size k from the uniform distribution on $[0, R(T)]$. (See Parzen (1962).) Hence the conditional expected running cost during the interval $[0, Y_{N(T)+1}]$ given $\{N(T) = k\}$ is

$$E((c + c_1 k) Y_N(T) + 1 - c_1 \sum_1^k Y_i \mid N(T) = k)$$

$$= (c + c_1 k)(T + M(T) - c_1 \sum_1^k E(R^{-1}(Z_i))$$

$$= (c + c_1 k)(T + M(T) - c_1 k E(R^{-1}(Z))$$

(where Z is uniformly distributed on [O, R(T)])

$$= (c + c_1 k)(T + M(T)) - c_1 k \int_0^T t\, r(t)\, dt / R(T)$$

$$= (c + c_1 k) M(T) + cT + c_1 k \int_0^T R(t)\,dt / R(T).$$

Theorem 1. The expected cost rate when it is subject to shocks of intensity rate $r(t)$ is

$$c(T) = c + [c_1 (M(T) R(T) + \int_0^T R(s)\,ds) + c_0] / (T + M(T)).$$

Proof. Lemma 1 makes it possible to calculate the expected running cost in the interval $[0, Y_{N(T)+1}]$:

$$\sum_{k=0}^{\infty} (c(T+M(T)) + c_1 k M(T) + c_1 k \int_0^T R(s)\,ds / R(T)) R^k(T) e^{-R(T)} / k!$$

$$= c(T + M(T)) + c_1 (M(T) + \int_0^T R(s)\,ds / R(T)) R(T)$$

$$= c(T + M(T)) + c_1 (M(T) R(T) + \int_0^T R(s)\,ds).$$

Hence, from the renewal reward theorem, the expected cost rate is

$$C(T) = [c(T + M(T)) + c_1 (M(T) R(T) + \int_0^T R(s)\,ds) + c_0] / (T + M(T)),$$

since the expected duration of a replacement interval is $T + M(T)$.

We seek the optimal value of T^*, *i.e.* the value of T minimizing $C(T)$.

Theorem 2. Let $H(T) = T + M(T) + TR(T) - \int_0^T R(s)\,ds$. Then if there exists a solution T^* satisfying $H(T^*) = c_0 / c_1$, it is the unique optimal solution. Otherwise, the optimal solution is $T^* = 0$ if $\mu > c_0 / c_1$ and $T^* = \infty$ if $\mu < c_0 / c_1$ and $\lim_{T \to \infty} H(T) < c_0 / c_1$, where $\mu = \int_0^\infty (1 - F(t))\,dt$.

Proof. First note that $M'(t) = M(T) r(T) - 1$. Hence,

$$C'(T) = [c_1 (M'(T) R(T) + r(T) M(T) + R(T)) (T + M(T))$$

$$- (1 + M'(T))(c_1 M(T) R(T) + \int_0^T R(s)\,ds + c_0)] / (T + M(T))^2$$

$$= M(T) r(T) [c_1 (T + M(T) + TR(T) - \int_0^T R(s)\,ds - c_0] / (T + M(T))^2$$

$$= M(T) r(T) (c_1 H(T) - c_0) / (T + M(T))^2.$$

Since $H'(T) = 1 + M'(T) + R(T) + Tr(T) - R(T) = r(T)(T + M(T)) > 0$, we know that $H(T)$ is a strictly increasing function of T. Hence if there exists a solution T^* satisfying $H(T^*) = c_0 / c_1$, it is the unique optimal solution. In fact, such a solution exists if $\mu < c_0 / c_1$ and $\lim_{T \to \infty} H(T) > c_0 / c_1$ since $\lim_{T \to 0} H(T) = M(0) = \mu$. The remaining part of the proof then easily follows.

We observe that the solution T^* is a function of the cost ratio c_0 / c_1 and not the individual values.

3. Examples

We now illustrate an example with an exponentially decreasing mean residual life function. This model was introduced by Muth(1977): it represents the case of strictly positive memory and has the advantage of being analytically tractable. Assume that

$$r(t) = \alpha \exp(\beta t) - \beta, \quad 0 < \beta < \alpha.$$

Then we have

$$R(t) = \alpha((\exp(\beta t) - 1)/\beta - \beta t.$$

Hence, we get

$$M(T) = \exp(-\beta T)/\alpha,$$

and

$$H(T) = e^{\beta T}(\alpha\beta T - 1)/\beta^2 + e^{\beta T}/\alpha - \beta T^2/2 + T + \alpha/\beta^2.$$

The equation $H(T) = c_0/c_1$ can now be solved by numerical methods. The following table contains the optimal value, T^*, for some values of α and c_0/c_1 with $\beta = \alpha/2$.

Table 1. Optimal Value of T

c_0/c_1 \ α	0.5	1
5	7.745	1.928
10	8.122	2.645
20	8.711	3.399
50	9.871	4.476
100	11.040	5.357

Table 2. Cost Comparison between Policy A and Policy B

c_0 \ α	0.5		1	
	Policy A	Policy B	Policy A	Policy B
5	4.025	4.327	3.279	3.350
10	4.937	5.192	5.182	5.223
20	6.574	6.772	8.242	8.266
50	10.725	10.851	15.512	15.523
100	16.521	16.605	25.443	25.448

From the table, the optimal value of T, T^*, increases with c_0/c_1. This is from the fact the if a finite solution exists for the equation $H(T) = c_0/c_1$, the solution should increase with c_0/c_1 since $H(T)$ is a strictly increasing function of T.

We now compare the proposed replacement policy (Policy A) with that of Boland and Proschan (Policy B, 1983), in which replacement is made at age T. Without loss of generality, we assume that

$c = 0$ and $c_1 = 1$. Table 2 summarizes the results.

From Table 2, we can see that the proposed policy is superior to that of Boland and Proschan (1983) in this example.

4. Remarks

A replacement policy for a system subject to shocks is proposed where each shock to the system increases the running cost. The proposed policy seems to be superior to that of Boland and Proschan, the proof of which is not given here. It will be interesting to extend these results to a case where there exist many types of shocks.

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